Mathematical Analysis G12MAN 2008/9

Assessed Coursework 2

Due 5 pm on Thursday 27 November 2008.

Unless otherwise specified, you must always justify your answers.

Throughout this coursework, d denotes a fixed positive integer.

Where you use standard results, state clearly which they are.

There are hand-in boxes in the Maths/Physics building and in the Pope building. You must fill out a yellow coursework cover sheet, staple it to your work, and date-stamp it using the machine provided, before placing your work in the box. Cover sheets are available next to the hand-in boxes.

Total marks obtainable 100. Each question is worth 25 marks.

1 Consider the following subset of $\mathbb{R}^2$:

$$S = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 4 \text{ and } x + y > 1\}.$$

(a) Draw a carefully labelled sketch of the set S. You should show your working, describe the key features of the set S, and indicate these key features clearly on your sketch. [15]

(b) Using your answer to (a) to help you, write down your answers to the following questions without further justification.

(i) Is the set S bounded? [2]

(ii) Is the set S open? [4]

(iii) Is the set S closed? [4]

2 Consider the three sequences of real numbers, (x_n), (y_n) and (z_n) defined as follows. For each $n \in \mathbb{N}$, set

$$x_n = n/4^n, \quad y_n = \frac{3n^2 + 2n - 3}{2n^2 + 3n + 1}, \quad \text{and} \quad z_n = 3^n/n!.$$

Define a sequence $(\boldsymbol{a}_n) \subseteq \mathbb{R}^3$ by

$$\boldsymbol{a}_n = (x_n, y_n, z_n) \quad \text{for each } n \in \mathbb{N}.$$

Determine, with justification, the limit of the sequence $(\boldsymbol{a}_n)$. [25 marks]

3 Determine, with justification, whether or not the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous.

(a) $f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ [10 marks]

(b) $g(x,y) = \begin{cases} \dfrac{x^2 y^5}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ [15 marks]

4 (a) Give, with justification, an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and two sequences (x_n), (y_n) of real numbers such that $|x_n - y_n| \to 0$ as $n \to \infty$, and yet

$$|f(x_n) - f(y_n)| \geq 1 \quad \text{for all } n \in \mathbb{N}\,.$$

[**Hint:** the sequences (x_n) and (y_n) can not be bounded here.] [10 marks]

(b) Let E be a non-empty, sequentially compact subset of $\mathbb{R}^d$, and let f be a continuous function from E to $\mathbb{R}$. Suppose that $(\boldsymbol{x}_n)$ and $(\boldsymbol{y}_n)$ are sequences of elements of E such that $\|\boldsymbol{x}_n - \boldsymbol{y}_n\| \to 0$ as $n \to \infty$. Prove that there exists an $n \in \mathbb{N}$ such that $|f(\boldsymbol{x}_n) - f(\boldsymbol{y}_n)| < 1$. [15 marks]