University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

G12MAN MATHEMATICAL ANALYSIS 2009-10

Solutions to Assessed Coursework 2

1. (a) The set S is the intersection of two relatively standard sets: $S = S_1 \cap S_2$, where

$$\begin{aligned} S_1 &= \{(x, y) \in \mathbb{R}^2 \mid (x+4)^2 + y^2 > 25\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - (-4, 0)\|^2 > 25\} \\ &= (\bar{B}_5((-4, 0))^c, \end{aligned}$$

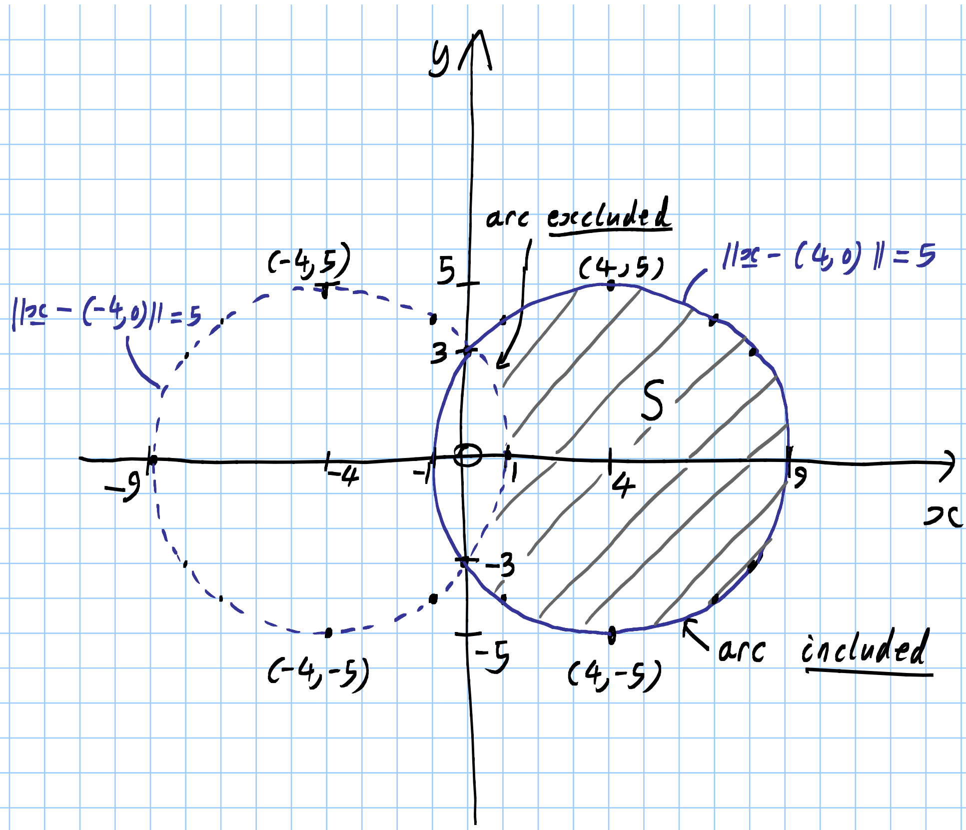
the **complement** of the closed disc with radius 5, centred on $(-4, 0)$.
Similarly,

$$\begin{aligned} S_2 &= \{(x, y) \in \mathbb{R}^2 \mid (x-4)^2 + y^2 \leq 25\} \\ &= \bar{B}_5((4, 0)), \end{aligned}$$

the closed disc of radius 5 centred on $(4, 0)$.
Note that S_1 excludes the points of the boundary circle, but S_2 includes the points of its boundary circle.
The use of the names S_1 and S_2 is optional here, assuming that equivalent justification is provided.
In fact, we have $S = \bar{B}_5((4, 0)) \backslash \bar{B}_5((-4, 0))$ here.
By observation (and Pythagoras) the two boundary circles meet at the points $(0, \pm 3)$.
The region S is the shaded region shown in the following diagram, excluding points of the dashed arc, but including points on the solid arc.

Reasoning + sketch [16 marks]

Note! To obtain full marks for this part of the question, you need to include sufficient reasoning here to show how you obtained your sketch. In particular, here, you should remember to state that the relevant curves are circles.
The sketch must show clearly all of the key features. In particular, the boundary curves should be clearly labelled with their equations, and it should be clear which parts of these curves are included/excluded from the region(s) under consideration.
You should usually also show the coordinates of a few key points, such as intersections of curves with each other or with the axes.



Sketch of $S \subseteq \mathbb{R}^2$ where

$$S = \bar{B}_5((4,0)) \setminus \bar{B}_5((-4,0))$$

S is shown shaded in grey, including the solid arc but excluding the dashed arc.

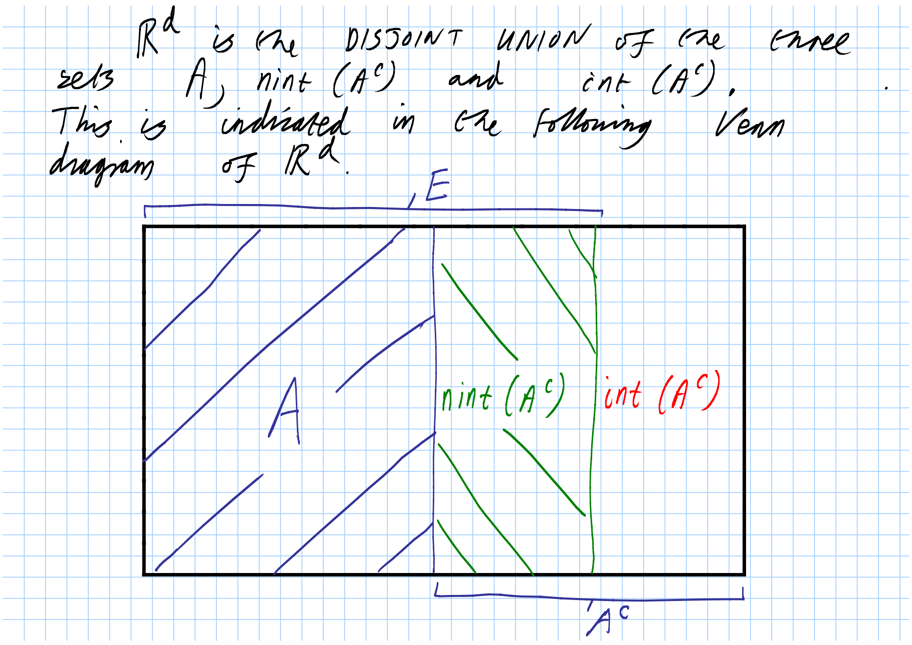
(b) (i) S is bounded. [3 marks]

(ii) S is **not** open. [3 marks]

(iii) S is **not** closed. [3 marks]

2. (a) We note that it is standard that $\mathbb{R}^d$ is the **pairwise disjoint** (i.e., non-overlapping) union of the three sets A, nint(A^c) and int(A^c).

This can be made very clear using the following (optional) Venn diagram, similar to a diagram from one of the examples classes.



Since $E = A \cup \text{nint}(A^c)$, it is now clear that $E^c = \text{int}(A^c)$, and it is standard that this is an **open** set.

Since E^c is open, E is closed. [10 marks]

(b) Given that F is closed and $A \subseteq F$, we have F^c is open and $F^c \subseteq A^c$. At this point I would accept a quote that int(A^c) is the "largest" open set in $\mathbb{R}^d$ which is $\subseteq A^c$, or else directly note that, since

$$F^c \subseteq A^c,$$

we have

$$\text{int}(F^c) \subseteq \text{int}(A^c),$$

and, as F^c is open, this gives

$$F^c \subseteq \text{int}\, A^c.$$

Thus

$$F \supseteq (\text{int}\, A^c)^c = (E^c)^c = E$$

(using part (a)) as required. [15 marks]

[**Note**: This will be revisited in G13MTS as "closure of A".]

3. (a) N.B. **No justification required**, but we include some additional comments.

nint $A = \{1, 5\}$. [Note: $A = [1, 5]$.]

nint $B = B = [2, 5]\backslash\mathbb{Q}$. [N.B. $\mathbb{Q}$ is **dense** in $\mathbb{R}$.]

nint $C = \emptyset$. [Here $C =]2, 3[\cup]3, 4[\cup]4, 5[$.]

nint $D = \{1\}$.

[6 marks]

(b)

A is not open,

B is not open,

C is open,

D is not open.

[4 marks]

(c)

A is closed,

B is not closed,

C is not closed,

D is closed.

[4 marks]

(d) A, B and C are bounded. D is unbounded. [4 marks]

(e) By the Heine–Borel theorem, a subset of $\mathbb{R}$ is sequentially compact if and only if it is both closed **and** bounded. From this, and parts (c) and (d):

A **is** sequentially compact, as it is both closed and bounded;
B is not sequentially compact because it is not closed;
C is not sequentially compact because it is not closed;
D is not sequentially compact because it is unbounded. [7 marks]

4. (a) This function f is discontinuous at $(0, 0)$, and so it is discontinuous. Consider $(x, y) \neq (0, 0)$ with $x = y^2 \neq 0$. Then

$$\begin{aligned} f(x, y) &= \frac{(y^2)^6 y^{20}}{(y^2)^{16} + y^{32}} \\ &= \frac{y^{32}}{2y^{32}} = \tfrac{1}{2} \\ &\not\to 0 \quad \text{as} \quad y \to 0. \end{aligned}$$

Thus, approaching the origin along the curve $x = y^2$ shows that

$$f(x, y) \not\to 0 = f(0, 0) \quad \text{as} \quad (x, y) \to (0, 0).$$

Thus f is discontinuous at $(0, 0)$, as claimed. [10 marks]

(b) The function g is continuous. For $(x, y) \neq (0, 0)$ we have $x^{16} + y^{32} \neq 0$, and so g is continuous away from $(0, 0)$ as it is a standard rational function of x and y. Since $g(0, 0) = 0$, it is now sufficient to prove that $g(x, y) \to 0$ as $(x, y) \to (0, 0)$. We do this by estimating $|g(x, y)|$.

For $(x, y) \neq (0, 0)$, we consider two cases.

Case I. $|x| \leq |y|^2$. In this case, since $(x, y) \neq (0, 0)$, we must have $y \neq 0$. We have

$$|x^6 y^{22}| = |x|^6 |y|^{22} \leq (|y^2|)^6 |y|^{22} = |y|^{34},$$

while

$$|x^{16} + y^{32}| = |x|^{16} + |y|^{32} \geq |y|^{32},$$

so

$$|g(x, y)| = \frac{|x^6 y^{22}|}{|x^{16} + y^{32}|} \leq \frac{|y|^{34}}{|y|^{32}} = |y|^2.$$

Case II. If case I does not hold, then we must have $|x| > |y|^2$, and so $x \neq 0$. Then certainly $|x| \geq |y|^2$, so

$$|x^6 y^{22}| = |x|^6 |y|^{22} \leq |x|^6 |x|^{11} = |x|^{17}$$

while

$$|x^{16} + y^{32}| = |x^{16}| + |y|^{32} \geq |x|^{16},$$

giving

$$|g(x, y)| = \frac{|x^6 y^{22}|}{|x^{16} + y^{32}|} \leq \frac{|x|^{17}}{|x|^{16}} = |x|.$$

Combining cases I and II, we see that, whenever $(x, y) \neq (0, 0)$, we have

$$0 \leq |g(x, y)| \leq |x| + |y|^2.$$

As $(x, y) \to (0, 0)$, RHS $\to 0 + 0 = 0$ by the algebra of limits, and so, by the sandwich theorem, $|g(x, y)|$, and hence also $g(x, y) \to 0$ as $(x, y) \to (0, 0)$.

[15 marks]