G12MAN: Mathematical Analysis, Module Syllabus

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This syllabus is intended to give you an overview of the **key concepts and results of the module**. It is **not**, however, a comprehensive list of the examinable contents of the module.

1. Introduction to $\mathbb{R}^d$

Some set theory. Standard inner product and Euclidean norm in $\mathbb{R}^d$. Triangle inequality.

2. Boundedness of subsets of $\mathbb{R}^d$

Examples of subsets of $\mathbb{R}^d$, including open balls and closed balls. Bounded sets and unbounded sets: definitions and examples, including bounded and unbounded d-cells.

3. Open subsets of $\mathbb{R}^d$

Interior points. Non-interior points **(non-standard terminology)**. The interior of a set. Open sets: definition, examples and non-examples. 'Open balls' really are open.

4. Topology of $\mathbb{R}^d$

Unions and intersections of open sets. Finite intersections of open sets are open. Finite unions and countable unions of open sets are open. Countable intersections of open sets need not be open. Closed sets: definition, examples and non-examples. Finite unions of closed sets are closed. Finite intersections and countable intersections of closed sets are closed.

5. Sequences in $\mathbb{R}^d$

Convergence of sequences in $\mathbb{R}^d$: definition, terminology and notation. **Non-standard terminology:** absorption of sequences by sets, and the stages by which sets absorb sequences. Convergence of a sequence in $\mathbb{R}^d$ in terms of the associated d sequences of co-ordinates. Sandwich theorem. Algebra of limits for sequences in $\mathbb{R}^d$. The sequence criterion for closedness of a subset of $\mathbb{R}^d$.

6. Subsequences and sequential compactness

The nested intervals principle in $\mathbb{R}$ and the nested d-cells principle in $\mathbb{R}^d$. Diameter of non-empty, bounded sets: definition and examples. Subsequences of sequences. Bolzano-Weierstrass theorem. Sequential compactness. Heine-Borel theorem (sequential compactness version), characterizing the sequentially compact subsets of $\mathbb{R}^d$ as those which are both closed and bounded.

7. Functions, limits and continuity

Functions of a single variable and of several variables. Function limits. Continuity and discontinuity of functions. Demonstrating discontinuity using various curves of approach.

8. Further theory of function limits and continuity

Algebra of limits for real-valued function limits. Sandwich theorem for real-valued function limits. Applications to showing that some function limits exist, using careful case-by-case analysis. Addition, scalar multiplication, product, modulus and quotient (avoiding division by 0) of continuous functions. Composition of continuous functions. Function limits and continuity in terms of ε and δ. Continuous images of sets: the continuous image of a sequentially compact set is sequentially compact. The boundedness theorem for continuous real-valued functions defined on non-empty, sequentially compact subsets of $\mathbb{R}^d$.

9. Sequences of Functions

Notation for sequences of functions. Pointwise convergence of sequences of functions: definition and examples. **Non-standard terminology and notation:** function balls and absorption of sequences of functions. Uniform convergence of sequences of functions: examples and non-examples. A uniform limit of continuous functions must be continuous, while a pointwise limit of continuous functions may be discontinuous. It is impossible for a sequence of unbounded functions to converge uniformly to a bounded function, and it is impossible for a sequence of bounded functions to converge uniformly to an unbounded function.

10. Rigorous differential calculus

Differentiability and differentiable functions. Examples and non-examples. Differentiability implies continuity, but continuity does not imply differentiability. The standard rules for differentiation: product rule, quotient rule and chain rule. Fermat's Theorem for Stationary Points, Rolle's Theorem and the Mean Value Theorem: statements, proofs and applications.

11. An introduction to Riemann integration

Partitions. Riemann upper and lower sums. Riemann upper and lower integrals. Riemann integrable functions: examples and non-examples. Continuous functions are Riemann integrable. The mean value theorem of integral calculus. The fundamental theorem of calculus.