Mathematical Analysis **G12MAN** **2009/10**

Feedback on Assessed Coursework 2

Common errors, pointers for improvement, and other comments

As for the first coursework, the results were generally good. The feedback below is based on a combination of comments passed to me by the postgraduate markers and what I saw on the scripts that I looked through.

1. (a) This part was mostly well done, with appropriate levels of working/justification shown, and clearly labelled sketches. The most common imperfections were: not showing working or justification; not describing the boundary curves as circles; not labelling the boundary curves with their equations (note here that the equations of the curves each involve an **equality**, and not an **inequality**); not (correctly) making it clear which parts of the curves were included/excluded; not noting other key features such as intersection points of curves with each other or the axes.

   (b) This part was mostly very well done, with answers matching the sketch from part (a).

2. Generally students found this to be the hardest of the four questions. Nevertheless, part(a) was mostly well done, with part (b) causing more problems.

   Some students attempted to solve one or both parts of this question using case by case analysis, e.g. first looking at the cases where A was closed, or A was open. However, as most sets are neither open nor closed (and some sets are both open and closed), looking at these special cases does not usually help in this type of question.

   (a) The key here was to justify convincingly the fact that $E^c = \mathrm{int}(A^c)$, and then to note that it is standard that the interior of a set is open. Most students did quite a good job here, but there were one or two things that could have been clearer.

   Note that a set is closed **if and only if** its complement is open. Some students noted that, if E is closed, then E^c is open. This is true, but in part (a) it is the **converse** to this statement that is more important: once you have shown that E^c is open, it follows from this that E is closed.

   Some students noted that $\mathbb{R}^d$ is the union of A, $\mathrm{nint}(A^c)$ and $\mathrm{int}(A^c)$, but did not clarify that this union is a (pairwise) **disjoint** union (the sets are non-overlapping). This fact is an important part of the reasoning.

   Other students applied de Morgan's rules correctly to say that

   $$E^c = (A \cup \mathrm{nint}(A^c))^c = A^c \cap (\mathrm{nint}(A^c))^c \,.$$

   However, in some cases, students then claimed that $(\mathrm{nint}(A^c))^c = \mathrm{int}(A^c)$: in fact

   $$(\mathrm{nint}(A^c))^c = \mathrm{int}(A^c) \cup ((A^c)^c)) = \mathrm{int}(A^c) \cup A.$$

   However, it is still true that

   $$A^c \cap ((\mathrm{nint}(A^c))^c) = A^c \setminus \mathrm{nint}(A^c) = \mathrm{int}(A^c)\,.$$

A more subtle error here was when students noted that $E^c \cap \operatorname{nint}(A^c) = \emptyset$, and then said

'Thus E^c has no non-interior points, and hence E^c is open.'

To illustrate the problem with this reasoning, consider the sets $X = [1, 2]$ and $Y = [0, 3] \subseteq \mathbb{R}$, so that $\operatorname{nint} X = \{1, 2\}$ and $\operatorname{nint} Y = \{0, 3\}$. Then $X \subseteq Y$ and $X \cap \operatorname{nint} Y = \emptyset$, but nevertheless $\operatorname{nint} X \neq \emptyset$. So, just because none of the non-interior points of Y are in X, it does not follow that $\operatorname{nint} X$ is empty: X has non-interior points of its own! Note that, in this example, X is a **strict** subset of $\operatorname{int} Y$. So, in this question, You really need to pin down the fact that E^c is the **whole** of the interior of A^c, and not just a subset of $\operatorname{int}(A^c)$.

(b) This part caused the most problems, but there were some excellent answers, either as in model answers or using proof by contradiction.

Some students were, perhaps, not quite sure what had to be proved here. Note that, unless A is closed, we will not have $E \subseteq A$ here. So there will often be **non-closed** sets F with $A \subseteq F$ but such that E is not a subset of F. (If A is not closed, then $F = A$ will be one example of this.) In view of this, we **must** use the fact that F is closed somewhere in the proof.

Some students claimed that $X \not\subseteq Y$ means the same thing as $X \cap Y = \emptyset$. However, these are two very different statements. For example, consider the sets $X = [1, 3]$ and $Y = [2, 4] \subseteq \mathbb{R}$. Then $X \not\subseteq Y$, but $X \cap Y \neq \emptyset$. On the other hand, **unless X is the empty set**, it is true that

$$X \cap Y = \emptyset \Rightarrow X \not\subseteq Y\,.$$

3. This question was very well done, with many students scoring close to full marks.

(a) Although no justification is required here, you usually need to determine what the sets actually are in order to answer this type of question. The most common mistakes were with set A (where some students believed that $3 \in \operatorname{nint} A$) and B (some students incorrectly believed that $\operatorname{nint} B = \emptyset$). Some students lost marks for using incorrect or unclear set notation. It is very important for you to get your set notation right, as I really need to **know** (without having to guess!) which sets you mean.

(b) This was well done. Note that full marks were obtained here provided that your answers agreed with your answers to part (a) (even if your answers to (a) were wrong).

(c) and (d) were very well done.

(e) Here I expected you to justify your answers by quoting the Heine–Borel theorem (a subset of $\mathbb{R}^d$ is sequentially compact if and only if it is both closed and bounded), and then using your answers to the earlier parts of the question. I expected you to note **here** which sets were both closed and bounded, **and which property or properties were missing for the remaining sets**. Many students omitted this last part of the justification, and just referred to the earlier parts of the question in a less specific way: this was not fully convincing. As a general rule, you should expect to write a little more when justification **is** required, as it is here, than when justification is not required.

Some students did not state the Heine–Borel theorem in a strong enough form: note that we need 'if and only if', and not just a one-way implication here.

4. This was mostly well done. The question is of a fairly standard type, and is particularly close to a question from one of the examples classes.

   (a) This was very well done, with most students demonstrating that there was a way to approach the origin without the function values tending to 0.

   Some students did not note that $f(0,0) = 0$, and some students were not careful enough about avoiding division by zero.

   (b) This part caused more problems, but there were many excellent answers.

   Some students lost a few marks by not observing that the function is clearly continuous away from the origin, as it is a standard rational function of x and y with (away from the origin) a denominator that does not take the value 0.

   For continuity at the origin, some students tried to use the cases $|y| \geq |x|$ and $|y| < |x|$ (or similar), and ran into trouble. Of these students, some incorrectly claimed that $|y|^{-4} \to 0$ as $y \to 0$. Others noticed that this was false, but then claimed (incorrectly) that the function was discontinuous. In fact, if you have a 'failed sandwich' of this type, then your argument is inconclusive: you need to try another argument in order to determine the answer.

   There were also a variety of minor mistakes in expression or in the manipulation of the inequalities involved.