G12MAN: Mathematical Analysis
Lecture by lecture summary of the module as taught in 2009-10

Lecturer: Dr J. Feinstein

Other staff/postgraduates involved: T. Blann, A. Camara, M. Johanesova, D. Nicks

Workshop 1: Module information and other documents issued. (All handouts are also available from the web.) Discussion of the nature of the module, including some examples showing why we sometimes need to be careful.
There will be more on this in the three optional special sessions on proof. See the Module Information page for details.
Revision quiz.

Lecture 1: **Chapter 1. Introduction to $\mathbb{R}^d$**
Revision/notation: sets, subsets, intersections and unions of finitely many sets. Set difference and complement. Cartesian products: definition, notation, examples and non-examples. Cartesian products of finitely many sets. Revision/notation: $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^d$. Vectors and points. Column vectors. Standard inner product (dot product). Euclidean norm $\|\cdot\|$.

Lecture 2: Properties of Euclidean norm, including triangle inequality and homogeneity. Euclidean distance $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ (from Pythagoras). Triangle inequality for Euclidean distance. Final comments on Chapter 1.
Chapter 2. Boundedness of subsets of $\mathbb{R}^d$.
Examples of subsets of $\mathbb{R}^d$, especially open balls and closed balls.

Lecture 3: Bounded sets: definition and brief discussion. Examples of bounded sets and unbounded sets. (See also intervals/d-cells below, and many further examples and results on question sheet 1.) Non-empty, bounded sets can be described as sets which have **finite diameter**. Bounded intervals in $\mathbb{R}$. Note in particular alternative notation $]a,b[$ for the open interval with endpoints a and b, to avoid confusion with the ordered pair (a,b). Half-open intervals. Most subsets of $\mathbb{R}$ are NOT intervals. The term half-open should **only** be used for (appropriate) intervals. Definitions of bounded d-cells in $\mathbb{R}^d$ (pre-2007-8 these were called **intervals** in $\mathbb{R}^d$): rectangles, cuboids, hyper-cuboids etc.

Lecture 4: Examples of bounded d-cells in $\mathbb{R}^d$. Unbounded d-cells in $\mathbb{R}^d$.
Chapter 3. Open subsets of $\mathbb{R}^d$
Interior points, non-interior points and interior of subsets of $\mathbb{R}^d$: definitions and examples.

Lecture 5: Careful justification of statements concerning interior and non-interior points. More examples.

Lecture 6: Open sets. Definitions of **open** and **not open** in terms of interior points/ non-interior points. Brief discussion of intervals in $\mathbb{R}$: open intervals are open, other types of non-empty interval are not open. Open balls are open (including brief discussion of why this needs a proof). Closed balls are not open. Both the empty set $\emptyset$ and $\mathbb{R}^d$ are open in $\mathbb{R}^d$. More examples and non-examples of open subsets of $\mathbb{R}^d$. Of the non-empty bounded d-cells (intervals, rectangles, cuboids etc.) in $\mathbb{R}^d$ discussed earlier, the only ones which are open are (unsurprisingly) the ones that we called 'open d-cells' before.

Lecture 7: **Chapter 4. Topology of $\mathbb{R}^d$**
Obvious but important fact: If $A \subseteq B$ then $\operatorname{int} A \subseteq \operatorname{int} B$, i.e., every interior point of A is also

an interior point of B.
A union or intersection of two open sets is again an open set. (An easy induction shows this holds also for finite unions and finite intersections.) Countable unions (unions of infinite sequences of sets). Countable unions of open sets are open. [So are uncountable unions! See G13MTS for more details.] Countable intersections of sets discussed. An example of a countable intersection of open sets which is not open.

Lecture 8: Closed sets: a set is closed if its complement is open.
Make sure that you know the difference between 'closed' and 'not open'!
Examples of closed sets. Sets which are both open and closed (clopen sets). Sets which are neither open nor closed. Because of de Morgan's laws, countable intersections of closed sets are closed and finite unions of closed sets are closed.
Chapter 5. Sequences in $\mathbb{R}^d$.
Sequence notation. Convergence of sequences in $\mathbb{R}^d$: definition, terminology and notation.

Lecture 9: Non-standard terminology: sets which absorb sequences, and the stages by which they absorb the sequences. Examples. The d sequences of coordinates associated with a sequence of vectors in $\mathbb{R}^d$: notation $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nd})$. Interpretation as a matrix with infinitely many rows and d columns: the rows are the vectors in the sequence, the columns are the sequences in $\mathbb{R}$ corresponding to the individual coordinates. Convergence in terms of convergence of these d individual coordinate sequences. Recalled: Sandwich Theorem in $\mathbb{R}$. Algebra of limits for sequences in $\mathbb{R}^d$. Brief discussion of the sequence criterion for closedness: informally this says that a set C is closed if and only if you can not 'escape' from C by taking the limit of a sequence of points from C.

Lecture 10: Example of a sequence in a subset of $\mathbb{R}$ such that the sequence converges in $\mathbb{R}$ to a limit which is outside the subset. Proof of the sequence criterion for closedness.
Chapter 6. Subsequences and Sequential Compactness
The nested intervals principle in $\mathbb{R}$ (recalled). The definition of diameter for non-empty, bounded of sets in $\mathbb{R}^d$. Examples. The nested d-cells principle in $\mathbb{R}^d$ (nested intervals in d dimensions). Discussion and sketches.

Lecture 11: Subsequences of sequences: definitions and examples. Sequences with infinitely many terms lying in $A \cup B$ must have a subsequence in A, or a subsequence in B (or both). Bolzano-Weierstrass theorem stated: every bounded sequence in $\mathbb{R}^d$ has at least one convergent subsequence. Some applications. Sketch of the proof of Bolzano–Weierstrass in the 1-dimensional case, using repeated bisection and nested intervals. Brief discussion of repeated quadrisection of rectangles, etc. Definition of sequential compactness. Discussion of the Heine–Borel Theorem (sequential compactness version), which characterizes the sequentially compact subsets of $\mathbb{R}^d$ as those which are both closed and bounded. Connections with sequence criterion for closedness and sequence criterion for boundedness. A sequence in $\mathbb{R}$ with no convergent subsequences.

Lecture 12: Proof of the Heine–Borel theorem. Examples of sequentially compact sets.
Chapter 7. Functions, limits and continuity
Functions: revision of domain, co-domain, restriction and image of functions. Functions from $\mathbb{R}^d$ to $\mathbb{R}^l$.

Lecture 13: Functions from D to $\mathbb{R}^l$ (where $D \subseteq \mathbb{R}^d$). Every function f taking values in $\mathbb{R}^l$ has l corresponding functions taking values in $\mathbb{R}$, one for each coordinate:

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_l(\mathbf{x}))\,.$$

Properties of last year's standard functions from $\mathbb{R}$ to $\mathbb{R}$ (continuity, differentiability) will be assumed without further proof in this module.
Limits and continuity of functions: definitions in terms of sequences, recalled for functions from $\mathbb{R}$ to $\mathbb{R}$ and generalized to functions defined on subsets of $\mathbb{R}^d$ and taking values in $\mathbb{R}^l$. Problems with isolated points. Polynomials and rational functions in several variables. Coordinate projections.

Lecture 14: Different ways to approach a point of $\mathbb{R}^2$, and corresponding ways to demonstrate that a function is discontinuous or a function limit does not exist. Properties of some examples of functions from $\mathbb{R}^2$ to $\mathbb{R}$. Comparison of continuity (also called 'joint continuity') and separate continuity. Separate continuity does NOT imply continuity. Example using different curves of approach to show that a function limit does not exist. For $D \subseteq \mathbb{R}^d$, a function f from D to $\mathbb{R}^l$ is continuous if and only if all of the l functions $f_1, f_2, \ldots, f_l$ (corresponding to the l coordinates of $f(\mathbf{x})$, as above) are continuous. Applications to checking that given functions are continuous.
Chapter 8. Further theory of function limits and continuity
New functions from old (pointwise addition, etc.). Algebra of limits for real-valued function limits: deduction from definitions, using the algebra of limits for sequences in $\mathbb{R}$.

Lecture 15: Sandwich theorem for real-valued function limits. Application to showing that a function limit exists for a specific function, using careful case-by-case analysis to estimate the modulus. New continuous functions from old. Pointwise operations for real-valued continuous functions: addition, scalar multiplication, product, modulus, quotient (avoiding division by 0). Composition of continuous functions. Equivalent definitions of function limits and continuity in terms of ε and δ.

Lecture 16: Example of bad behaviour of discontinuous functions. Application of the ε-δ characterization of continuity: continuous real-valued functions that are positive at a point remain positive near the point. Images of continuous functions: the continuous image of a sequentially compact set is sequentially compact. Bad behaviour discussed of continuous images of various other types of sets (open sets, closed sets, bounded sets).

Lecture 17: Student evaluation forms issued. Every non-empty sequentially compact subset of $\mathbb{R}$ has both a maximum and minimum element. The boundedness theorem for continuous real-valued functions defined on non-empty, sequentially compact subsets of $\mathbb{R}^d$.
Chapter 9. Sequences of functions
Motivation. Pointwise convergence: definition and examples.

Lecture 18: Function balls: definition and illustrative sketch. Definition of uniform convergence in terms of function balls and absorption of sequences of functions. Examples illustrating the differences between pointwise and uniform convergence.

Lecture 19: More examples. Uniform convergence is better than pointwise convergence: a uniform limit of continuous functions must be continuous, while a pointwise limit of continuous functions may be discontinuous. It is impossible for a sequence of unbounded functions to converge uniformly to a bounded function, and it is impossible for a sequence of bounded functions to converge uniformly to an unbounded function.

Chapter 10. Rigorous differential calculus.

Definition of differentiability for real-valued functions on open subsets of $\mathbb{R}$. Differentiability implies continuity, but the converse is untrue as shown by the function $f(x) = |x|$. Weierstrass's example of a function which is continuous everywhere but differentiable nowhere on $\mathbb{R}$. Brief discussion/revision of the standard rules for differentiation from G11CAL (product rule, quotient rule, chain rule). See books for rigorous proofs (the chain rule needs care). **From now on, you may assume that all the standard differentiable functions (polynomials, exponential, logarithmic, trigonometric, etc.) you met in G11CAL have the derivatives and properties discussed there.**

Lecture 20: Fermat's Theorem, Rolle's Theorem and the Mean Value Theorem (all proved in full) and their applications, e.g. to monotonicity of functions and estimates of function values.

Lecture 21: **Chapter 11. An introduction to Riemann integration**

The proofs of the lemmas and theorems concerning Riemann integration are not 'examinable as bookwork' in 2009-10 and you will not be asked to reproduce proofs of these in full in the January 2010 examination. However, you ARE expected to know the definitions and the statements of the results, and to know how to apply these, e.g., to investigate examples.

Motivation. Partitions. Riemann lower and upper sums (approximation using rectangles). Riemann lower and upper integrals. Riemann integrable functions. Examples of functions which are/are not Riemann integrable. Continuous functions are Riemann integrable. Fundamental Theorem of Calculus. Mean Value Theorem of Integral Calculus.