Mathematical Analysis G12MAN 2007/8

Assessed Coursework 2

Due 5 pm on Thursday 29 November 2007.

Unless otherwise specified, you must always justify your answers.

Throughout this coursework, d denotes a fixed positive integer.

Where you use standard results, state clearly which they are.

There are hand-in boxes in the Maths/Physics building and in the Pope building. You must fill out a coursework cover sheet, staple it to your work, and date-stamp it using the machine provided, before placing your work in the box. Cover sheets are available next to the hand-in boxes.

Total marks obtainable 100. Each question is worth 25 marks.

1 For each of the following sets, determine whether or not they are (i) open, (ii) closed (as subsets of $\mathbb{R}$ or $\mathbb{R}^2$ as indicated).

For each set you must give **separate** answers to (i) and (ii).

You should justify your answers briefly but convincingly, for example by sketching each set and its complement, and stating which points (if any) of the relevant sets are clearly non-interior points.

(a) $A = [0, 2] \cup]3, 6[\subseteq \mathbb{R}$; [6 marks]

(b) $B = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R}$; [10 marks]

(c) $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \leq 4\} \subseteq \mathbb{R}^2$. [9 marks]

2 Let $E \subseteq \mathbb{R}^d$. Quoting appropriate standard results to support your answer, prove that the following statements are equivalent, i.e., each implies the other. [It is probably best to prove **separately** that (a) $\Rightarrow$ (b) and $\neg$(a) $\Rightarrow$ $\neg$(b).]

(a) E is bounded;

(b) every sequence $(x_n) \subseteq E$ has a subsequence which converges in $\mathbb{R}^d$.

[Note in (b) that the limit need not be in E.] [25 marks]

3 Give specific examples, with brief justification, of the following.

(a) A bounded subset of $\mathbb{R}^2$ which is not sequentially compact. [5 marks]

(b) A closed subset of $\mathbb{R}^2$ which is not sequentially compact. [5 marks]

(c) An unbounded sequence in $\mathbb{R}$ which has a convergent subsequence. [5 marks]

(d) A strictly decreasing sequence of positive real numbers which is absorbed by the interval $[0, 10^{-6}]$, but which does not converge to 0. [5 marks]

(e) A sequence $(\boldsymbol{x}_n) \subseteq \mathbb{R}^2$ such that $(\boldsymbol{x}_n)$ is **divergent** (i.e. does not converge) but the sequence of real numbers $(\|\boldsymbol{x}_n\|)_{n=1}^{\infty}$ is **convergent**. [5 marks]

4 Determine, with justification, whether or not the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous.

(a) $$f(x,y) = \begin{cases} \dfrac{x^4 y}{x^8 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$ [10 marks]

(b) $$g(x,y) = \begin{cases} \dfrac{x^8 y}{x^8 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$ [15 marks]