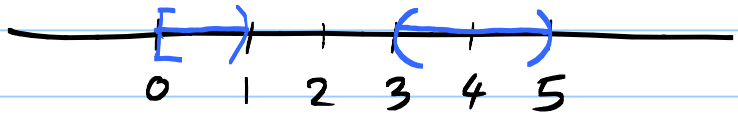
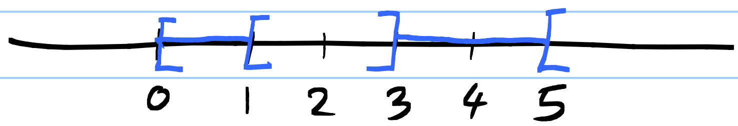
SOLUTIONS

1 (a)



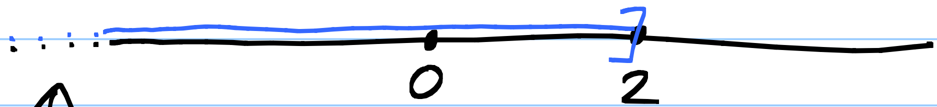
or



[I will use the first version in lectures.]

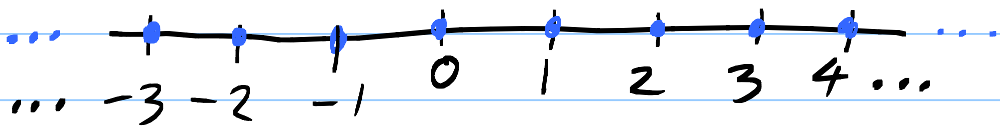
Other correct and clear styles of sketch also accepted here and below.

(b)



Some indication of infinite extent.

(c)



Points of set indicated as blue dots.

(d) [Many possible methods. Here is one.]

We have $x^4 \le 8x$

$\iff x^4 - 8x \le 0$

$\iff x(x^3-8) \le 0.$

The factors have sign changes when $x = 0$, or when $x^3-8 = 2$, i.e., $x=2$.

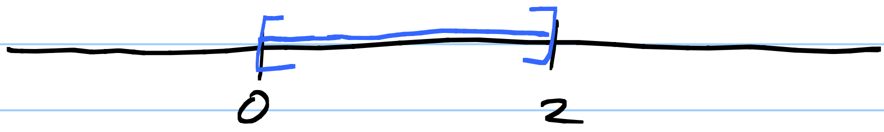
<u>Sign chart</u>

	$x < 0$	$0 < x < 2$	$x > 2$
x	$-$	$+$	$+$
x^3-8	$-$	$-$	$+$
$x(x^3-8)$	$+$	$-$	$+$

Thus $x(x^3-8) < 0 \iff 0 < x < 2$

and $x(x^3-8) \le 0 \iff 0 \le x \le 2.$

Thus $D = [0, 2]$.



OPTIONAL COMMENT

2. [From the diagrams it is clear that] A and D are bounded, while B and C are unbounded.
[No justification required.]

3. Since A is bounded, there is an $R > 0$ with

$$A \subseteq \overline{B}_R(\underline{0}).$$

Since B is bounded, there is an $R' > 0$ with $B \subseteq \overline{B}_{R'}(\underline{0})$.

Set $R'' = \max(R, R')$. Then

$$A \cup B \subseteq B_R(\underline{0}) \cup B_{R'}(\underline{0}) = B_{R''}(\underline{0}),$$

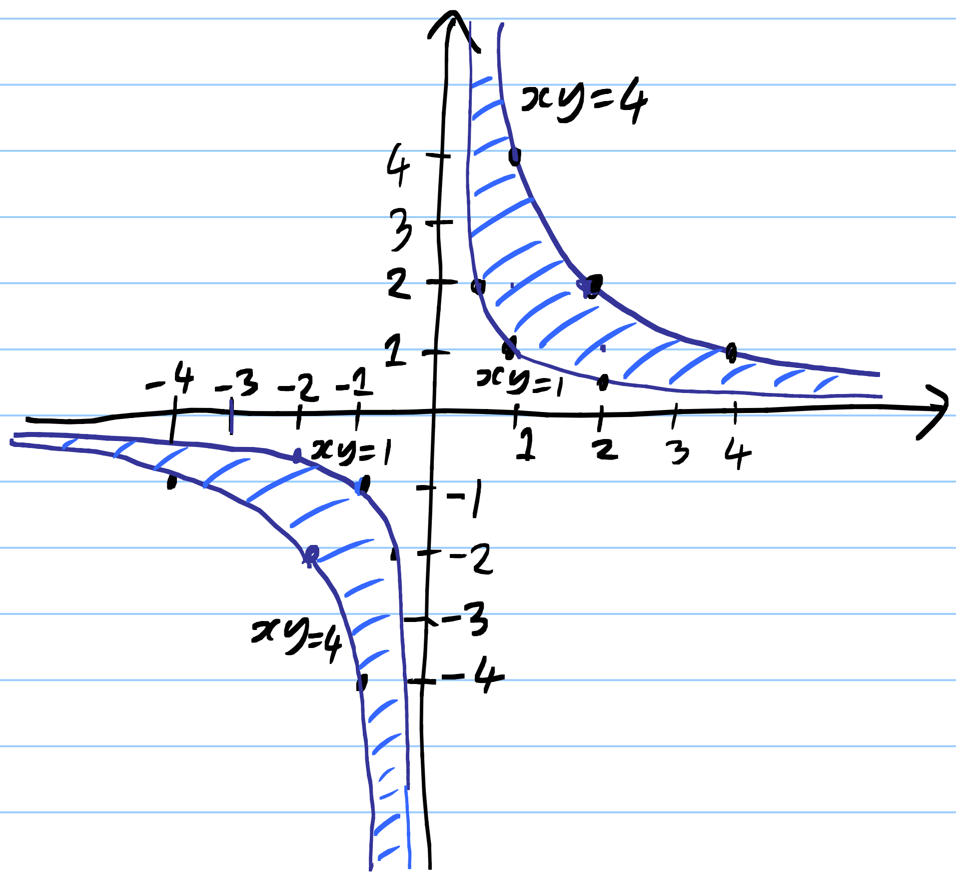
and so $A \cup B$ is bounded.

Illustrative diagrams are optional here.

4. (a) This set consists of two unbounded regions whose boundaries are given by the hyperbolae

$xy = 1, \qquad xy = 4.$ These

boundary curves are included. The set S is the shaded region shown below.



I would expect the sketch to be clearly labelled, and to show at least a few points such as (1,1), (2,2), (−1,−1) and (−2,−2).

(b) (i) Clearly the set is <u>unbounded</u>.

(ii) The set of interior points of S is the set of points in S which are strictly between the two hyperbolae, i.e., $\{(x,y) \in \mathbb{R}^2 \mid 1 < xy < 4\}$.

(iii) The set of non-interior points of S is the set of <u>points of S</u> which lie on the boundary hyperbolae. In this case this includes the whole of both hyperbolae, so the answer is

$$\{(x,y) \in \mathbb{R}^2 \mid xy = 1 \text{ or } xy = 4\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid xy = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid xy = 4\}.$$

5. This is false. For example, we may take

$$A = \mathbb{Q}, \quad B = \mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}.$$

Then $\operatorname{int} A = \operatorname{int} B = \emptyset$ (standard from notes) but

$$A \cup B = \mathbb{R},$$

so

$$\operatorname{int}(A \cup B) = \operatorname{int}(\mathbb{R}) = \mathbb{R}$$
$$\neq \operatorname{int} A \cup \operatorname{int} B = \emptyset.$$

[There are many other examples, e.g. $A = [0,1]$, $B = [1,2]$.]