Regularity conditions for Banach function algebras

Dr J. F. Feinstein

University of Nottingham

June 2009

1 Useful sources

A very useful text for the material in this mini-course is the book

Banach Algebras and Automatic Continuity

by H. Garth Dales, London Mathematical Society Monographs, New Series, Volume 24, The Clarendon Press, Oxford, 2000.

In particular, many of the examples and conditions discussed here may be found in Chapter 4 of that book.

We shall refer to this book throughout as **the book of Dales**.

Most of my e-prints are available from

www.maths.nott.ac.uk/personal/jff/Papers

Several of my research and teaching presentations are available from

www.maths.nott.ac.uk/personal/jff/Beamer

2 Introduction to normed algebras and Banach algebras

2.1 Some problems to think about

Those who have seen much of this introductory material before may wish to think about some of the following problems.

We shall return to these problems at suitable points in this course.

Problem 2.1.1 (Easy using standard theory!) It is standard that the set of all rational functions (quotients of polynomials) with complex coefficients is a field: this is a special case of the "field of fractions" of an integral domain.

Question: Is there an algebra norm on this field (regarded as an algebra over $\mathbb{C}$)?

Problem 2.1.2 (Very hard!) Does there exist a pair of sequences (λ_n), (a_n) of non-zero complex numbers such that

(i) no two of the a_n are equal,

(ii) $\sum_{n=1}^{\infty} |\lambda_n| < \infty$,

(iii) $|a_n| < 2$ for all $n \in \mathbb{N}$, and yet,

(iv) for all $z \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \lambda_n \exp(a_n z) = 0?$$

Gap to fill in

Problem 2.1.3 Denote by $C[0,1]$ the “trivial” uniform algebra of all continuous, complex-valued functions on $[0,1]$.

(i) **(Very hard!)** Give an example of a proper, uniformly closed subalgebra A of $C[0,1]$ such that A contains the constant functions and separates the points of $[0,1]$.

In other words, give an example of a non-trivial uniform algebra on $[0,1]$.

(ii) **(Impossible?)** Is there an example of an algebra A as in (i) with the additional property that the only non-zero, multiplicative linear functionals on A are the evaluations at points of $[0,1]$?

In other words, is there a non-trivial uniform algebra whose character space is $[0,1]$?

(This is a famous open problem of Gelfand.)

2.2 Revision of basic definitions

Banach algebras may be thought of as Banach spaces with multiplication (in a sense made more formal below).

The additional structure provided by the multiplication gives the theory of Banach algebras a rather different flavour from the more general theory of Banach spaces.

Banach algebras may be real or complex. However the theory of complex Banach algebras is richer, and so this is what we will focus on.

Banach algebras may be commutative or non-commutative. We will focus mainly on commutative Banach algebras.

A key example of a commutative Banach algebra is $C_{\mathbb{C}}(X)$, the algebra of continuous, complex-valued functions on a compact, Hausdorff space, with the usual pointwise operations and with the uniform norm.

A typical non-commutative Banach algebra is $B(E)$, the algebra of all bounded linear operators from E to E for some Banach space E (of dimension at least 2!), with the usual vector space structure and operator norm, and with product given by composition of operators.

In particular, for a Hilbert space H, $B(H)$ and its subalgebras are of major interest.

Just as the theory of Banach algebras should not be regarded as part of the theory of Banach spaces, the theory of $B(H)$ and its subalgebras has its own flavour and a vast literature, including the quite distinct study of C^*-algebras and von Neumann algebras.

Definition 2.2.1 A **complex algebra** is a complex vector space A which is a ring with respect to an associative multiplication which is also a bilinear map, i.e., the distributive laws hold and, for all $\alpha \in \mathbb{C}$ and a and b in A, we have

$$(\alpha a)b = a(\alpha b) = \alpha(ab)\,.$$

The complex algebra A **has an identity** if there exists an element $e \neq 0 \in A$ such that, for all $a \in A$, we have $ea = ae = a$.

Real algebras are defined similarly.

Note that, if A has an identity e, then this identity is unique, so we may call e **the identity** of A.

We will often denote the identity by 1 rather than e, assuming that the context ensures that there is no ambiguity.

The assumption above that $e \neq 0$ means that we do not count 0 as an identity in the trivial algebra $\{0\}$.

From now on, all algebras will be assumed to be complex algebras, so we will be defining **complex** normed algebras and Banach algebras.

Definition 2.2.2 A **(complex) normed algebra** is a pair $(A, \|\cdot\|)$ where A is a complex algebra and $\|\cdot\|$ is a norm on A which is **sub-multiplicative**, i.e., for all a and b in A, we have

$$\|ab\| \leq \|a\|\|b\|.$$

A normed algebra A is **unital** if it has an identity 1 **and** $\|1\| = 1$.

A **Banach algebra** (or **complete normed algebra**) is a normed algebra which is complete as a normed space.

Notes.

- In our usual way, we will often call A itself a normed algebra, if there is no ambiguity in the norm used on A.
- We will mostly be interested in commutative, unital Banach algebras.
- The condition that the norm on A be sub-multiplicative is only slightly stronger than the requirement that multiplication be jointly continuous (or equivalently, that multiplication be a 'bounded bilinear map') from $A \times A \to A$.

  In fact (**easy exercise**) if we have $\|ab\| \leq C\|a\|\|b\|$, for some constant $C > 1$, then we can easily find an equivalent norm on A which is actually sub-multiplicative.
- If a normed algebra $(A, \|\cdot\|)$ has an identity 1 such that $\|1\| \neq 1$, then we may again define another norm $|||\cdot|||$ on A as follows:

$$|||a||| = \sup\{\|ab\| : b \in \bar{B}_A(0,1)\}\,.$$

As another **exercise**, you may check that $|||\cdot|||$ is equivalent to $\|\cdot\|$, that $|||\cdot|||$ is sub-multiplicative and that $|||1||| = 1$.

So the assumption that $\|1\| = 1$ is a convenience, rather than a topological restriction.

- Every subalgebra of a normed algebra is a normed algebra, and every closed subalgebra of a Banach algebra is a Banach algebra.

Examples.

(1) Let E be a complex Banach space of dimension > 1. Then $(B(E), \|\cdot\|_{\text{op}})$ is a non-commutative, unital Banach algebra, where the product on $B(E)$ is composition of operators.

(2) Let X be a non-empty, compact, Hausdorff topological space. Then $C_{\mathbb{C}}(X)$ is a commutative, unital Banach algebra with pointwise operations and the uniform norm $|\cdot|_X$: recall that

$$|f|_X = \sup\{|f(x)| : x \in X\}.$$

As we are focussing on complex algebras, **from now on we will denote $C_{\mathbb{C}}(X)$ by $C(X)$.**

(3) With notation as in (2), every (uniformly) closed subalgebra of $C(X)$ is also a commutative Banach algebra with respect to the uniform norm $|\cdot|_X$.

A **uniform algebra on** X is a closed subalgebra A of $C(X)$ which contains all the constant functions and which **separates the points of** X, i.e., whenever x and y are in X with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$.

Every uniform algebra on X is, of course, a commutative, unital Banach algebra, with respect to the uniform norm.

Also, by Urysohn's Lemma, $C(X)$ itself **does** separate the points of X, and so $C(X)$ is a uniform algebra on X.

Gap to fill in

(4) In particular, let X be a non-empty, compact subset of $\mathbb{C}$.

Consider the following subalgebras of $C(X)$:

$A(X)$ is the set of those functions in $C(X)$ which are analytic (holomorphic) on the interior of X;

$P_0(X)$ is the set of restrictions to X of polynomial functions with complex coefficients;

$R_0(X)$ is the set of restrictions to X of rational functions with complex coefficients whose poles (if any) lie off X (so
$R_0(X) = \{p/q : p, q \in P_0(X), 0 \notin q(X)\}$).

It is easy to see that these subalgebras contain the constant functions and separate the points of X.

Indeed the polynomial function Z, also called the **co-ordinate functional**, defined by $Z(\lambda) = \lambda$ $(\lambda \in \mathbb{C})$, clearly separates the points of X by itself, and is in all of these algebras.

The algebra $A(X)$ is closed in $C(X)$, because uniform limits of analytic functions are analytic (see books for details), and so $A(X)$ is a uniform algebra on X.

The algebras $P_0(X)$ and $R_0(X)$ are usually not closed in $C(X)$ (**exercise:** investigate this).

We obtain uniform algebras on X by taking their (uniform) closures: $P(X)$ is the closure of $P_0(X)$ and $R(X)$ is the closure of $R_0(X)$.

The functions in $P(X)$ are those which may be **uniformly approximated** on X by polynomials, and the functions in $R(X)$ are those which may be **uniformly approximated** on X by rational functions with poles off X.

We have $P(X) \subseteq R(X) \subseteq A(X) \subseteq C(X)$.

Gap to fill in

(5) The Banach space $C^1[0,1]$ of once continuously differentiable complex-valued functions on $[0,1]$ is a subalgebra of $C[0,1]$.

It is not uniformly closed, and so it is not a uniform algebra on $[0,1]$.

However, it is a Banach algebra when given its own norm, $\|f\| = |f|_X + |f'|_X$ (where $X = [0,1]$).

This is a typical example of a **Banach function algebra**.

Warning! In the literature, some authors call uniform algebras **function algebras**.

Gap to fill in

(6) Every complex normed space E can be made into a (non-unital) normed algebra by defining the trivial multiplication $xy = 0$ $(x, y \in E)$.

Thus every complex Banach space is also a Banach algebra.

(7) The following standard construction can be used to 'unitize' Banach algebras.

Let $(A, \|\cdot\|_A)$ be a complex Banach algebra **without** an identity.

We may form a Banach algebra $A^{\#}$, called the **standard unitization** of A, as follows.

As a vector space, $A^{\#} = A \oplus \mathbb{C}$.

This becomes a Banach space when given the norm

$$\|(a, \alpha)\| = \|a\|_A + |\alpha| \quad (a \in A, \alpha \in \mathbb{C})\,.$$

We can then make $A^{\#}$ into a unital Banach algebra using the following multiplication: for a and b in A and α and β in $\mathbb{C}$, we define

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)\,.$$

Exercise. Check the details of these claims.

(What is the identity element of $A^{\#}$?)

Show also that $A^{\#}$ is commutative if and only if A is commutative.

We can combine this construction with (6) to give examples of commutative, unital Banach algebras which have many non-zero elements whose squares are 0.

This never happens, of course, for our algebras of functions.

2.3 Characters and the character space for commutative Banach algebras

In this section we will focus on commutative algebras. However, many of the definitions and results are valid (with some minor modifications) in the non-commutative setting too: see books for details.

Definition 2.3.1 Let A be a commutative algebra with identity 1.

An element $a \in A$ is **invertible** if there exists $b \in A$ with $ab = 1$.

In this case the element b is unique: b is then called the **inverse** of a, and is denoted by a^{-1}.

The set of invertible elements of A is denoted by $\operatorname{Inv} A$.

Notes.

- For non-commutative algebras, we would insist that both $ab = 1$ and $ba = 1$.
- For invertible elements a, it is clear that a^{-1} is invertible and $(a^{-1})^{-1} = a$.

- With multiplication as in A, $\mathrm{Inv}\, A$ is a group with identity 1, and the map $a \mapsto a^{-1}$ is a bijection from $\mathrm{Inv}\, A$ to itself.

- Let X be a non-empty, compact, Hausdorff topological space. Then

$$\mathrm{Inv}\, C(X) = \{f \in C(X) : 0 \notin f(X)\}\,.$$

  If X is a non-empty, compact subset of $\mathbb{C}$ then the same is true for $R(X)$ and $A(X)$, but not for $P(X)$ unless X has 'no holes' (see books for more details on this).

Gap to fill in

Theorem 2.3.2 Let A be a commutative, unital Banach algebra, and let $x \in A$ with $\|x\| < 1$. Then $1 - x$ is invertible, and

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots .$$

Thus whenever $a \in A$ with $\|a - 1\| < 1$, we have $a \in \operatorname{Inv} A$.

In other words, the open ball in A centred on the identity element and with radius 1 is a subset of $\operatorname{Inv} A$.

We now investigate characters and the character space of commutative, unital Banach algebras.

Definition 2.3.3 Let A be a commutative algebra.

A **character** on A is a **non-zero, multiplicative linear functional** on A, i.e. a non-zero linear functional $\phi : A \to \mathbb{C}$ satisfying $\phi(ab) = \phi(a)\phi(b)$ $(a, b \in A)$.

The set of all characters on A is called the **character space** of A, and is denoted by Φ_A.

Notes.

- Suppose that A has an identity, 1. Then it is elementary to show that $\phi(1) = 1$ for all $\phi \in \Phi_A$.

  It is also easy to show, in this case, that Φ_A is a closed subset of the product space $\mathbb{C}^A$.

- If A is an algebra of complex-valued functions on a non-empty set X (with pointwise operations), then, for every $x \in X$, there is an **evaluation character** at x, denoted by $\hat{x}$, defined by $\hat{x}(f) = f(x)$ $(f \in A)$.

  In general there may also be many other characters on A.

  However, it turns out that in the case of $C(X)$ (for compact, Hausdorff X) there are no others.

For non-empty compact subsets X of $\mathbb{C}$ (also known as **compact plane sets**), the same is true for $R(X)$ and for $A(X)$, but not for $P(X)$ unless X has 'no holes'.

See books for details: $A(X)$ is rather hard!

Gap to fill in

The following is the most basic of the 'automatic continuity' results concerning Banach algebras.

Theorem 2.3.4 Let A be a commutative, unital Banach algebra.

Then every character ϕ on A is continuous, with $\|\phi\| = 1$.

Using this and the Banach-Alaoglu Theorem, we obtain the following important corollary.

Corollary 2.3.5 Let A be a commutative, unital Banach algebra.

Then Φ_A is a weak-* compact subset of A^*.

The relative (i.e. subspace) weak-* topology on Φ_A is called the **Gelfand topology**.

The Gelfand topology is the weakest topology on Φ_A such that, for all $a \in A$, the map $\phi \mapsto \phi(a)$ is continuous.

We may restate the above corollary as follows:

Φ_A is a compact, Hausdorff topological space with respect to the Gelfand topology.

By default, we will always use the Gelfand topology on Φ_A.

In fact, every commutative, unital Banach algebra has at least one character: we will return to this later.

We conclude this section by recalling the definition of the **Gelfand transform**.

Definition 2.3.6 Let A be a commutative, unital Banach algebra. Then the **Gelfand transform** is the map from A to $C(\Phi_A)$ defined by $a \mapsto \hat{a}$, where $\hat{a}(\phi) = \phi(a)$ $(a \in A, \phi \in \Phi_A)$.

The **Gelfand transform of** A is the set $\widehat{A} = \{\hat{a} : a \in A\}$.

2.4 Semisimple, commutative, unital Banach algebras

You probably already know various definitions of the term semisimple.

We give our definition in terms of characters.

See books for the equivalence of this and the usual algebraic definition, in the setting of commutative, unital Banach algebras.

Definition 2.4.1 Let A be a commutative, unital Banach algebra. Then A is semisimple if

$$\bigcap_{\phi \in \Phi_A} \ker \phi = \{0\},$$

i.e., for every non-zero $a \in A$, there exists a character ϕ on A with $\phi(a) \neq 0$.

Notes.

- Every unital Banach algebra of functions on a set X is semisimple.

  This may be seen immediately by considering just the evaluation characters at points of X.

- Conversely, every semisimple, commutative, unital Banach algebra A is isomorphic (as an algebra) to a subalgebra of $C(\Phi_A)$.

  Indeed A is semisimple if and only if the Gelfand transform is injective, in which case A is isomorphic to its Gelfand transform $\widehat{A}$.

We have implicitly assumed above the obvious notions of **algebra homomorphism** (a multiplicative linear map) and **algebra isomorphism** (a bijective algebra homomorphism).

We also need the notion of a **unital** algebra homomorphism.

Definition 2.4.2 Let A and B be commutative, unital Banach algebras, with identities 1_A and 1_B respectively.

A **unital** algebra homomorphism from A to B is an algebra homomorphism $T : A \to B$ such that $T(1_A) = 1_B$.

The following remarkable Automatic Continuity results are true concerning semisimple, commutative, unital Banach algebras.

The first result concerns the automatic continuity of homomorphisms.

Theorem 2.4.3 Let A and B be commutative, unital Banach algebras, and suppose that B is semisimple. Then every unital algebra homomorphism from A to B is automatically continuous.

The next result (a corollary) is a **uniqueness of norm** result.

Corollary 2.4.4 Let $(A, \|\cdot\|)$ be a semisimple, commutative, unital Banach algebra.

Suppose that $\|\cdot\|'$ is another norm on A such that $(A, \|\cdot\|')$ is a commutative unital Banach algebra. Then the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Gap to fill in

2.5 Resolvent and spectrum

We begin this section with some further facts concerning the invertible group $\mathrm{Inv}\, A$ of a commutative, unital Banach algebra A.

For convenience, we will use the abbreviation CBA for commutative Banach algebra.

Theorem 2.5.1 Let A be a unital CBA. Then the following facts hold:

(a) $\mathrm{Inv}\, A$ is open in A;

(b) the map $a \mapsto a^{-1}$ is a homeomorphism from $\mathrm{Inv}\, A$ to itself.

We now define the **resolvent set** and the **spectrum** for an element of a commutative algebra with identity.

Definition 2.5.2 Let A be a commutative algebra with identity and let $x \in A$.

Then the **spectrum of x in A**, $\sigma_A(x)$ (or $\sigma(x)$ if the algebra under consideration is unambiguous) is defined by

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \notin \mathrm{Inv}\, A\}\,.$$

The **resolvent set of x in A**, $\rho_A(x)$, is the complement of the spectrum, i.e.,

$$\rho_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \in \operatorname{Inv} A\}\,.$$

Notes. We are mainly interested in the case of CBA's. So, let A be a unital CBA.

- Since $\operatorname{Inv} A$ is open in A, it follows easily that the resolvent set is open in $\mathbb{C}$, and hence that the spectrum is closed in $\mathbb{C}$.
- We have $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$. Thus the spectrum is always a compact subset of $\mathbb{C}$.

The next result is a consequence of Liouville's Theorem (complex analysis).

See books for the elegant details.

Proposition 2.5.3 Let A be a commutative, unital normed algebra. Then, for all $x \in A$, $\sigma(x) \neq \emptyset$.

The next result shows that $\mathbb{C}$ itself is the only complex normed algebra which is a field.

Theorem 2.5.4 (Gelfand-Mazur) Let A be a commutative, unital normed algebra.

Suppose that $\operatorname{Inv} A = A \setminus \{0\}$. Then $A = \operatorname{lin}\{1\}$, and A is isometrically isomorphic to $\mathbb{C}$.

Thus none of the many non-trivial extension fields of $\mathbb{C}$ can be given a (complex) algebra norm.

Gap to fill in

We conclude this section with the definition and formula for the spectral radius.

Definition 2.5.5 Let A be a commutative, unital Banach algebra, and let $x \in A$.

We define the **spectral radius** of x (in A), $\nu_A(x)$, by

$$\nu_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}.$$

The following result is the famous **spectral radius formula**.

Theorem 2.5.6 Let A be a commutative, unital Banach algebra, and let $x \in A$. Then

$$\nu_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}.$$

Gap to fill in

2.6 Maximal ideals and characters

When working with algebras rather than just rings, we insist that an **ideal** in the algebra must be a linear subspace, in addition to being an ideal in the ring theory sense.

Fortunately, this makes no difference in the case where the algebra has an identity.

Definition 2.6.1 Let A be a commutative algebra with identity. Then a **proper ideal** in A is an ideal I in A such that $I \neq A$.

A **maximal ideal** in A is a maximal proper ideal in A (with respect to set inclusion).

An easy Zorn's Lemma argument shows that every proper ideal I in A is contained in at least one maximal ideal in A.

We conclude this introductory chapter with a standard result which connects up the concepts discussed so far.

Theorem 2.6.2 Let A be a unital CBA. Then the following hold.

(a) Every maximal ideal in A is closed.

(b) For every character $\phi \in \Phi_A$, $\ker \phi$ is a maximal ideal in A.

(c) Conversely, every maximal ideal in A is the kernel of a unique character on A. In particular, every maximal ideal has codimension 1 in A.

(d) For every $x \in A$, we have

$$\sigma_A(x) = \{\phi(x) : \phi \in \Phi_A\}.$$

(e) The character space Φ_A is non-empty, and, for all $x \in A$, the spectrum of x is equal to the image of the Gelfand transform of x, i.e.,

$$\sigma_A(x) = \hat{x}(\Phi_A).$$

Gap to fill in

Gap to fill in

3 Banach function algebras

3.1 Preliminary definitions and results

We saw in Chapter 1, via the Gelfand transform, that semisimple, unital commutative Banach algebras are (essentially) the same thing as Banach algebras of continuous functions on compact Hausdorff spaces.

Given a non-unital Banach algebra A, we know how to form the standard unitization $A^{\#}$.

Given a non-semisimple, unital, commutative Banach algebra A, the standard way to make it semisimple is to quotient out by the **Jacobson radical**.

The Jacobson radical, J, of the unital CBA A, is the intersection of all of the maximal ideals in A.

(In the non-commutative setting, you should use maximal one-sided ideals.)

With the **quotient norm**, A/J is then a unital, semisimple CBA.

From now on, we will work mostly with unital, semisimple CBA's.

However, many of the definitions and results generalize in standard ways to all CBA's using the comments above.

For example, **most** of the named conditions we discuss hold for a given non-unital CBA A if and only if they hold for its standard unitization $A^{\#}$.

Indeed, in many cases, this can be used as the definition in the non-unital case.

Gap to fill in

In our terminology, a **compact space** is a non-empty, compact, Hausdorff topological space.

Definition 3.1.1 Let X be a compact space.

A **normed function algebra** on X is a normed algebra $(A, \|\cdot\|)$ such that A is a subalgebra of $C(X)$, such that A contains the constants and separates the points of X, and such that, for all $f \in A$, we have $\|f\| \geq |f|_X$.

A normed function algebra $(A, \|\cdot\|)$ is a **Banach function algebra** on X if it is complete.

As in Chapter 1, a **uniform algebra** on X is a Banach function algebra A on X such that the norm on A is the uniform norm $|\cdot|_X$.

Of course, in the case where $(A, \|\cdot\|)$ is a Banach algebra and a subalgebra of $C(X)$, it is automatic that $\|f\| \geq |f|_X$ for all $f \in A$.

Let A be a Banach function algebra on a compact space X.

We define

$$\varepsilon_x : f \mapsto f(x)\,, \quad A \to \mathbb{C}\,,$$

for each $x \in X$.

Then $\varepsilon_x \in \Phi_A$, and the map $x \mapsto \varepsilon_x$, $X \to \Phi_A$, is a continuous embedding.

We say that A is **natural** (on X) if this map is surjective.

A typical example of a uniform algebra which is not natural is $P(X)$ when X is a compact plane set with at least one 'hole'.

Gap to fill in

It is standard that $C(X)$ is always natural on X.

More generally, we state a well-known result on naturality for 'full' subalgebras of $C(X)$. Here, a subalgebra A of $C(X)$ is **full** if $\mathrm{Inv}\, A = A \cap \mathrm{Inv}\, C(X)$.

Proposition 3.1.2 Let A be a Banach function algebra on a compact space X.

If A is self-adjoint, and A is a full subalgebra of $C(X)$, then A is natural on X.

Gap to fill in

Using the Gelfand transform, every unital, semisimple CBA can be regarded as a natural Banach function algebra on its character space (which is compact in this setting).

From now on, this is what we mean when we talk about **a Banach function algebra A on Φ_A**.

The well-known Banach algebra H^∞ (bounded analytic functions on the open unit disc) is a uniform algebra on its character space. However, this character space is quite complicated in nature.

Gap to fill in

The following useful test for naturality, due to T. Honary (1990), is not very well known.

Proposition 3.1.3 Let X be a compact space, and let $(A, \|\cdot\|)$ be a Banach function algebra on X, with uniform closure B. Then A is natural on X if and only if both of the following conditions hold:

(a) B is natural on X;

(b) $\lim_{n\to\infty} \|f^n\|^{1/n} = 1$ for each $f \in A$ with $|f|_X = 1$.

The proof of this result is an **exercise**.

We can use either of the two tests for naturality above to prove that the Banach function algebra $C^1[0,1]$ is natural on $[0,1]$.

(There are, of course, many easy proofs of this fact.)

Gap to fill in

3.2 Two deep theorems

The next pair of theorems are very easy to state, but their proofs require the deep theory of several complex variables.

In our terminology, a **clopen** set is a set which is both open and closed.

Theorem 3.2.1 (Shilov Idempotent Theorem) Let A be a Banach function algebra on Φ_A.

Suppose that E is a clopen subset of Φ_A.

Then the characteristic function of E, χ_E, is in A.

In other words, every idempotent in $C(\Phi_A)$ is automatically in A.

Gap to fill in

Theorem 3.2.2 (Arens–Royden)

Let A be a Banach function algebra on Φ_A, and let $f \in C(\Phi_A)$.

If $\exp(f) \in A$, then $f \in A$.

In other words, whenever a function in A has a continuous logarithm defined on Φ_A, then that continuous logarithm must itself be in A.

Gap to fill in

3.3 Regularity and normality

Definition 3.3.1 Let A be a Banach function algebra on a compact space X. Then A is **regular on** X if, for each proper, closed subset E of X and each $x \in X \setminus E$, there exists $f \in A$ with $f(x) = 1$ and $f(y) = 0 \ (y \in E)$;

A is **regular** if it is regular on Φ_A;

A is **normal on** X if, for each proper, closed subset E of X and each compact subset F of $X \setminus E$, there exists $f \in A$ with $f(x) = 1 \ (x \in F)$ and $f(y) = 0 \ (y \in E)$;

A is **normal** if it is normal on Φ_A.

Gap to fill in

Urysohn's lemma tells us that the uniform algebra $C(X)$ is always normal, and hence also regular.

For any compact plane set X with non-empty interior, the uniform algebras $P(X)$, $R(X)$ and $A(X)$ are not regular (hence not normal).

It is easy to check that the Banach function algebra $C^1[0,1]$ is normal (and hence regular).

We will meet many more examples of regular (and normal) Banach function algebras later.

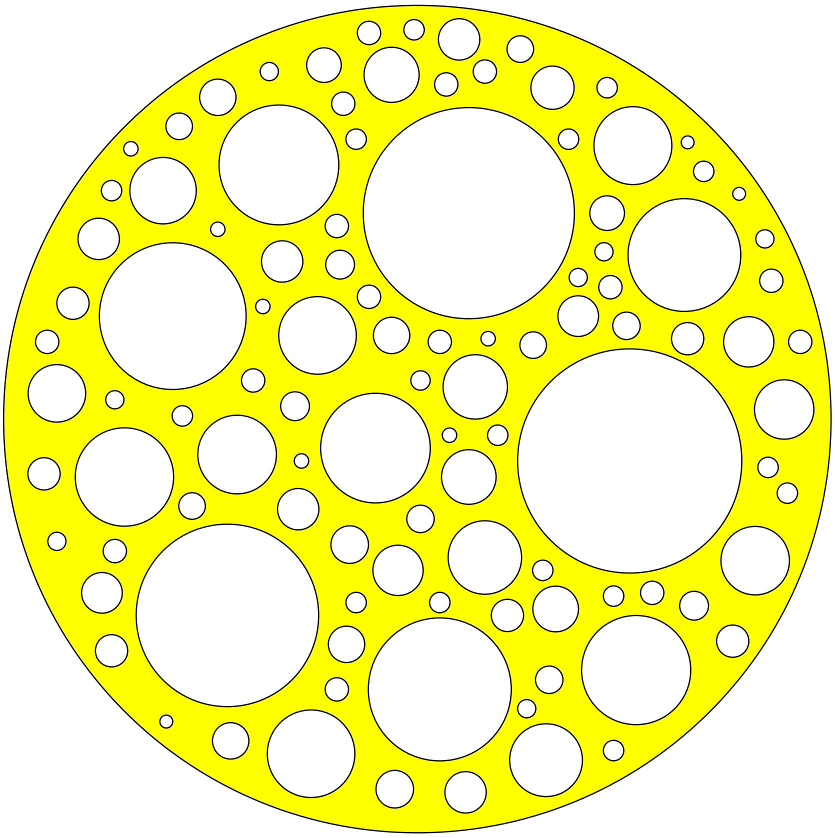
Gap to fill in

Normal uniform algebras are less common.

The first non-trivial, normal uniform algebra was constructed by McKissick in 1963.

His example was $R(X)$ for a suitable **Swiss cheese** set $X \subseteq \mathbb{C}$.

Gap to fill in



A Swiss cheese set with non-overlapping holes

Gap to fill in

Definition 3.3.2 Let A be a Banach function algebra on a compact space X, and let I be an ideal in A.

We define the **hull** of I, $h(I)$, to be the intersection of the zero sets of the functions in I, i.e.,

$$h(I) = \bigcap_{f \in I} f^{-1}(\{0\})\,.$$

Now let E be a subset of X.

We define the **kernel** of E, $I(E)$ (sometimes denoted by $k(E)$), to be the closed ideal in A,

$$\{f \in A : f(E) \subseteq \{0\}\}\,,$$

consisting of those functions in A which vanish identically on E.

Suppose that A is natural on X, so that we may regard X as equal to Φ_A.

In this setting, since every proper ideal is contained in a maximal ideal, it follows that **no proper ideal in A has empty hull**.

This is false without the assumption of naturality.

The connection with regularity is given by the following standard result.

Proposition 3.3.3 Let A be a Banach function algebra on a compact space X.

Then A is regular on X if and only if, for every closed subset E of X, we have $h(I(E)) = E$.

More generally, we can define the **hull-kernel topology** on X using $E \mapsto h(I(E))$ as a closure operation.

Let A be a Banach function algebra on Φ_A.

Then the hull-kernel topology on Φ_A is weaker than the Gelfand topology, and the topologies agree if and only if A is regular.

Otherwise, the hull-kernel topology is non-Hausdorff.

In the next result, it is not enough to assume **regularity on** X, unless you know that your algebra is natural on X.

Theorem 3.3.4 Every regular Banach function algebra is normal.

Gap to fill in

The definition of normality is reminiscent of Urysohn's lemma.

However, in view of the following result of Bade and Curtis, we can not hope to maintain control of the norms of the functions f separating closed sets from each other.

This result (and its proof) may be found as Theorem 4.1.19 in the book of Dales.

Theorem 3.3.5 Let $(A, \|\cdot\|)$ be a Banach function algebra on a compact space X, let $M > 0$, and let $c \in (0, 1/2)$.

Suppose that, for every pair of disjoint closed subsets E and F of X, there is an $f \in A$ with $\|f\| \leq M$ and such that $|f(x)| < c \ (x \in E)$ and $|f(x) - 1| < c \ (x \in F)$.

Then $A = C(X)$.

3.4 Some examples of Banach function algebras

We have already discussed the uniform algebras $C(X)$ (for compact spaces X) and $P(X)$, $R(X)$ and $A(X)$ (for compact plane sets X).

Obviously, we can also work with compact subsets of $\mathbb{C}^N$ for $N > 1$.

We also mentioned the Banach function algebra $C^1[0,1]$.

Similarly, we can define $C^n[0,1]$, the Banach function algebra of n-times continuously differentiable complex-valued functions on the interval $[0,1]$.

However, the algebra $C^\infty[0,1]$ of infinitely-differentiable functions on $[0,1]$ is not a Banach function algebra (no matter which norm you try).

Gap to fill in

Other well-known Banach function algebras include the Lipschitz algebras $\mathrm{Lip}\,_\alpha(X)$ and $\mathrm{lip}\,_\alpha(X)$ for a compact metric space X, and Fourier algebras $A(\Gamma) = L^1(G)$ where G and Γ are mutually dual, locally compact, abelian groups.

These Banach function algebras are regular (and, indeed, have many stronger properties).

See Sections 4.4 and 4.5 of the book of Dales for many more details concerning these algebras.

Gap to fill in

There are also many useful examples of Banach function algebras on $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, the one-point compactification of $\mathbb{N}$.

Here, $C(\mathbb{N}_\infty)$ may be identified with the algebra of all convergent complex sequences.

Exercise.

(i) **(You may wish to quote the Shilov Idempotent Theorem!)** Prove that every natural Banach function algebra on $\mathbb{N}_\infty$ is regular.

(ii) Give an example of a Banach function algebra on $\mathbb{N}_\infty$ that is not regular on $\mathbb{N}_\infty$.

Gap to fill in

For $1 \leq p < \infty$, the usual sequence spaces ℓ^p, are (non-unital) subalgebras of $C(\mathbb{N}_\infty)$, and these subalgebras may be unitized to give natural, Banach function algebras on $\mathbb{N}_\infty$.

Another interesting family of Banach function algebras on $\mathbb{N}_\infty$ is the following.

Let $\alpha = (\alpha_n)_{n=1}^\infty$ be a sequence of positive real numbers. We define A_α by

$$A_\alpha = \left\{ f \in C(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)| < \infty \right\}.$$

It is easy to see that A_α is a subalgebra of $C(\mathbb{N}_\infty)$, and that A_α is a Banach function algebra, where the norm of a function $f \in A_\alpha$ is given by

$$\|f\| = \|f\|_\infty + \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)|.$$

It is also easy to check that the character space of A_α is just $\mathbb{N}_\infty$.

We will frequently return to these examples in the remaining chapters.

3.5 Regularity and decomposable operators

We conclude this chapter by discussing an interesting connection, due to M. Neumann (1992), between regularity and the theory of **decomposable operators**.

Definition 3.5.1 Let E be a Banach space, and let $T \in B(E)$. Then T is **decomposable** if, for every open cover $\{U, V\}$ of $\mathbb{C}$, there are closed, invariant subspaces F and G for T such that $E = F + G$, $\sigma(T|_F) \subseteq U$ and $\sigma(T|_G) \subseteq V$.

Gap to fill in

Let A be a Banach function algebra. For each $f \in A$, we denote by T_f the multiplication operator in $B(A)$ defined by $g \mapsto fg$.

Neumann investigated the decomposability of these multiplication operators, and proved the following.

Proposition 3.5.2 (M. Neumann, 1992) Let A be a Banach function algebra on Φ_A.

For each $f \in A$, the multiplication operator T_f is decomposable if and only if the function f is continuous when Φ_A is given the **hull-kernel** topology.

Thus the Banach function algebra A is regular if and only if, for all $f \in A$, the multiplication operator T_f is decomposable.

Gap to fill in

4 Regularity conditions and their implications

4.1 Regularity and the structure of ideals

Recall the notions of the **hull** $h(I)$ of an ideal I (the intersection of the zero sets of the functions in I) and the **kernel** $I(E)$ of a set E (defined again below).

We now restrict attention to closed sets E.

Let A be a Banach function algebra on a compact space X.

For each closed set $E \subseteq X$ we define a pair of ideals, $I(E)$ (as before) and $J(E)$ by

$$I(E) = \{f \in A : f(E) \subseteq \{0\}\,\}$$

and

$$J(E) = \{f \in A : E \subseteq \operatorname{int}\left(f^{-1}(\{0\})\right)\}\,.$$

Thus the functions in $I(E)$ are 0 at all points of E, while each function $f \in J(E)$ is 0 at all points of some neighbourhood of E: the neighbourhood depends on f.

For $x \in X$, we denote the maximal ideal $I(\{x\})$ by M_x, and we write J_x for $J(\{x\})$.

Even without regularity, we always have $h(M_x) = \{x\}$.

However, an easy compactness argument shows that A is regular on X if and only if, for all $x \in X$, we have $h(J_x) = \{x\}$.

Gap to fill in

The next result shows a close connection between regularity and the ideal structure.

This result (and its proof) may be found as Proposition 4.1.20 in the book of Dales.

Proposition 4.1.1 Let A be a regular Banach function algebra on Φ_A, and let E be a closed subset of Φ_A. Then:

(i) $J(E)$ is the minimum ideal in A whose hull is E;

(ii) $\overline{J(E)}$ is the minimum **closed** ideal in A whose hull is E;

(iii) $I(E)$ is the maximum ideal in A whose hull is E;

(iv) for each ideal I in A, we have

$$J(S) \subseteq I \subseteq I(S)\,,$$

where $S = h(I)$.

In this setting, the quotient algebras $I(E)/\overline{J(E)}$ provide a good source of **radical Banach algebras**.

However, this topic is beyond the scope of this course.

Recall that a **prime ideal** in a commutative, complex algebra A is a **proper** ideal P in A with the property that $A \setminus P$ is multiplicatively closed.

Note that every maximal ideal in a unital CBA is a closed prime ideal, and that the trivial ideal $\{0\}$ is prime if and only if the algebra is an integral domain.

Gap to fill in

The next standard result places restrictions on the possible prime ideals in regular algebras.

Proposition 4.1.2 Let A be a regular Banach function algebra on Φ_A, and let P be a prime ideal in A. Then $h(P)$ has only one element, x say, and we have

$$J_x \subseteq P \subseteq M_x \, .$$

Gap to fill in

4.2 Spectral synthesis and strong regularity

Definition 4.2.1 Let A be a Banach function algebra on a compact space X, and let E be a closed subset of X.

We say that E is a **set of synthesis** for A if $\overline{J(E)} = I(E)$.

Now let $x \in X$. We say that A is **strongly regular at** x if $\{x\}$ is a set of synthesis for A, i.e., if $\overline{J_x} = M_x$.

The Banach function algebra is **strongly regular** on X, if it is strongly regular at all points of X.

Spectral synthesis holds for A (or A **has spectral synthesis**) on X if every closed subset of X is a set of synthesis for A.

Clearly, whenever spectral synthesis holds for A on X, then A is strongly regular on X.

It is also easy to see that, if A is strongly regular on X, then A is regular on X.

However, an elegant argument of Mortini shows that, whenever A is strongly regular on X, then A is natural on X.

Thus every such algebra is natural and regular (and hence normal), and we may, without ambiguity, omit the 'on X' above.

Gap to fill in

4.3 Examples

At this point, let us see where our previous examples fit in.

- For every compact space X, $C(X)$ has spectral synthesis, and is strongly regular (etc.).
- For each $n \in \mathbb{N}$, the Banach function algebra $C^n[0,1]$ is **not** strongly regular.

Gap to fill in

- (Sherbert, 1964) For every compact metric space X and $\alpha \in (0,1)$, the 'little' Lipschitz algebras $\mathrm{lip}_{\alpha}(X)$ have spectral synthesis.

  However, for $\alpha \in (0,1]$, the **only** closed sets which are of synthesis for $\mathrm{Lip}_{\alpha}(X)$ are the **clopen** subsets of X.

  Thus, unless the compact space X is discrete (and hence finite), $\mathrm{Lip}_{\alpha}(X)$ is not strongly regular, and does not have spectral synthesis.

- Let Γ be a locally compact, abelian group. Then the Fourier algebra $A(\Gamma)$ is strongly regular.

  However, it is a famous theorem of Malliavin that $A(\Gamma)$ does not have spectral synthesis unless Γ is discrete.

Gap to fill in

- All of the examples of Banach sequence algebras discussed earlier have spectral synthesis, and hence are strongly regular.

  In general, regular Banach sequence algebras need not be strongly regular.

- Let X be a compact plane set such that $R(X) \neq C(X)$.

  It is not known whether $R(X)$ can be strongly regular, or whether $R(X)$ can have spectral synthesis.

- It is open whether or not a non-trivial uniform algebra can have spectral synthesis.

- There are non-trivial, strongly regular uniform algebras. The first known examples were due to Feinstein (1992).

Gap to fill in

4.4 Spectral synthesis and closed ideals

Let A be a Banach function algebra on a compact space X.

Suppose that A is strongly regular. Then A is natural on X, A is regular, and, for all $x \in X$, the maximal ideal M_x is the unique closed ideal in A whose hull is $\{x\}$.

In particular, every closed prime ideal in A is a maximal ideal.

Now suppose, further, that A has spectral synthesis.

In this case, we have a complete description of the closed ideals in A: they are precisely the kernels of the closed subsets of A.

Gap to fill in

Conversely, a Banach function algebra A on Φ_A has spectral synthesis if and only if A is regular, and the only closed ideals in A are the kernels $I(E)$ of closed sets $E \subseteq \Phi_A$.

It appears to be open whether or not the regularity assumption here is redundant:

Suppose that the only closed ideals in A are the kernels $I(E)$ of closed sets $E \subseteq \Phi_A$.

Does it follow that A is regular, and hence that A has spectral synthesis?

Gap to fill in

4.5 Bounded approximate identities and strong Ditkin algebras

We begin this section by recalling the definition of bounded approximate identity.

Definition 4.5.1 Let B be a commutative Banach algebra (usually **without** identity).

A **bounded approximate identity** in B is a bounded net $(e_\alpha) \subseteq B$ such that, for all $b \in B$, the net $(e_\alpha b)$ converges to b in B.

Gap to fill in

We now introduce the strong Ditkin algebras.

Definition 4.5.2 Let A be a Banach function algebra. Then A is a **strong Ditkin algebra** if A is strongly regular, and every maximal ideal in A has a bounded approximate identity.

There is an intermediate condition between being **strongly regular** and being a **strong Ditkin algebra**, namely being a 'Ditkin algebra'.

We shall not discuss this condition in detail here, but those interested may wish to consult Chapter 4 of the book of Dales.

Gap to fill in

From now on we use the standard abbreviation **b.a.i.** for bounded approximate identity.

Returning to our standard examples, we have the following.

- For every compact space X, $C(X)$ is a strong Ditkin algebra.
- Consider the algebra $\mathrm{lip}_{\alpha}(X)$, where X is a compact metric space and $\alpha \in (0, 1)$.

  Let $x \in X$. Then M_x has a b.a.i. if and only if x is an isolated point of X.

  Thus, if X is infinite, then $\mathrm{lip}_{\alpha}(X)$ is a Banach function algebra which has spectral synthesis, but is **not** a strong Ditkin algebra.
- For every locally compact abelian group Γ, the Fourier algebra $A(\Gamma)$ **is** a strong Ditkin algebra.

  Thus, if Γ is not discrete, then $A(\Gamma)$ is a strong Ditkin algebra which does not have spectral synthesis.
- For $p \in [1, \infty)$, the (standard unitizations of the) Banach sequence algebras ℓ_p are not strong Ditkin algebras.

Gap to fill in

- The situation for the algebras A_α depends on the sequence α.

  Recall that

$$A_\alpha = \left\{ f \in C(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)| < \infty \right\}.$$

  where $\alpha = (\alpha_n)$ is a sequence of positive real numbers.

  In this case, A_α is a strong Ditkin algebra if and only if (α_n) has a bounded subsequence.

  Thus A_α fails to be a strong Ditkin algebra if and only if $\alpha_n \to +\infty$ as $n \to \infty$.

Gap to fill in

For a compact plane set X, $R(X)$ is a strong Ditkin algebra if and only if $R(X) = C(X)$.

However, there are non-trivial, strong Ditkin uniform algebras.

Strongly regular uniform algebras need not be strong Ditkin algebras.

Gap to fill in

4.6 Automatic continuity and Banach extensions for strong Ditkin algebras

We now recall the definitions of Banach A-bimodules, derivations, and intertwining maps, which play an important role in the cohomology theory of Banach algebras.

Definition 4.6.1 Let A be a commutative Banach algebra.

A **Banach A-bimodule** is an A-bimodule E which is also a Banach space such that both of the bilinear maps $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto x \cdot a$ are jointly continuous from $A \times E$ to E.

Given a Banach A-bimodule E, a linear map D from A to E is a **derivation** if, for all $a, b \in A$ we have

$$D(ab) = a \cdot D(b) + D(a) \cdot b\,.$$

A linear map T from A to E is an **intertwining map** if, for all $a \in A$, both of the maps $b \mapsto T(ab) - a \cdot T(b)$ and $b \mapsto T(ba) - T(b) \cdot a$ are continuous from A to E.

Clearly, every derivation into a Banach A-bimodule is an intertwining map.

Gap to fill in

A powerful automatic continuity result holds for strong Ditkin algebras.

This result, and its proof, may be found as Corollary 5.3.5 in the book of Dales.

Theorem 4.6.2 Let A be a strong Ditkin algebra and let E be a Banach A-bimodule.

Then every intertwining map from A to E is automatically continuous.

In particular, every derivation from a strong Ditkin algebra A into a Banach A-bimodule is automatically continuous.

Gap to fill in

Another important cohomological property of strong Ditkin algebras concerns the splitting of Banach extensions.

The following theorem, and its proof, may be found as Theorem 5.4.41 in the book of Dales.

Theorem 4.6.3 Let A be a strong Ditkin algebra or a C^*-algebra.

Then every Banach extension of A which splits also splits strongly.

Gap to fill in

5 Gelfand's problem revisited, mathematical curiosities and some open problems

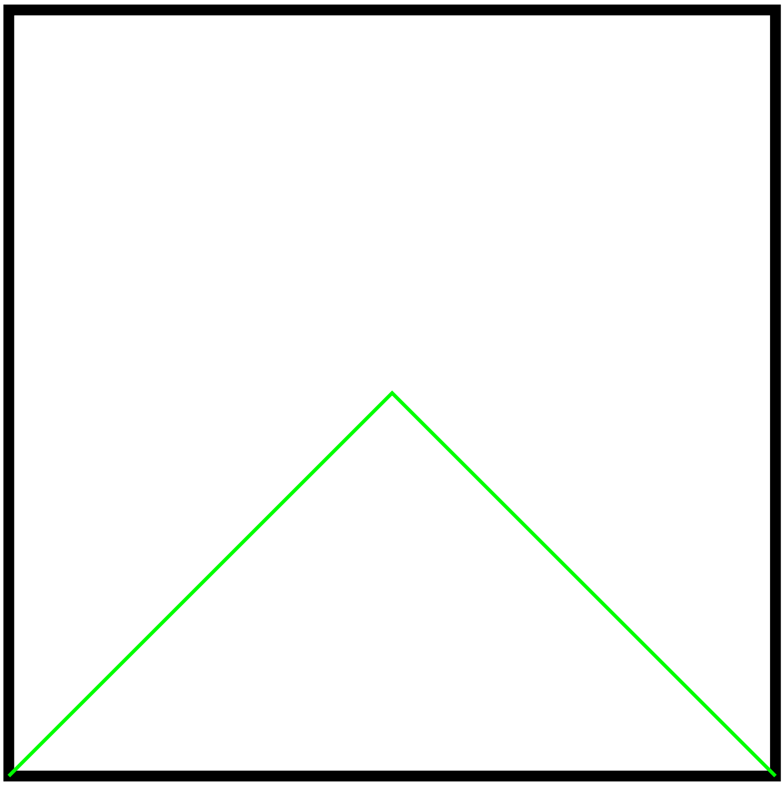
5.1 Space-filling curves and Jordan arcs with positive area

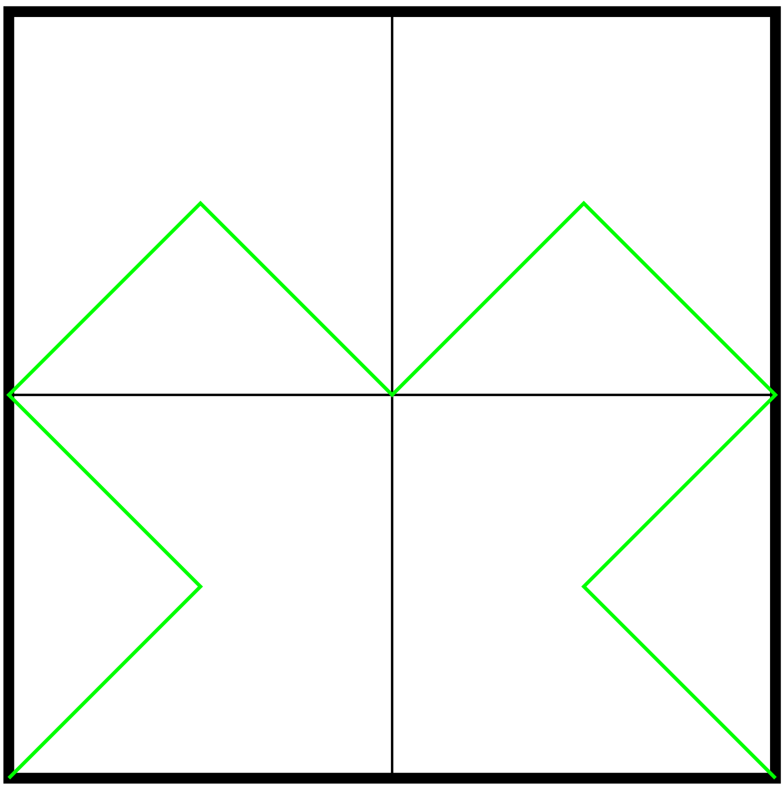
In order to give examples of non-trivial uniform algebras on the interval, we shall need the fact that there are Jordan arcs in $\mathbb{C}$ which have positive area.

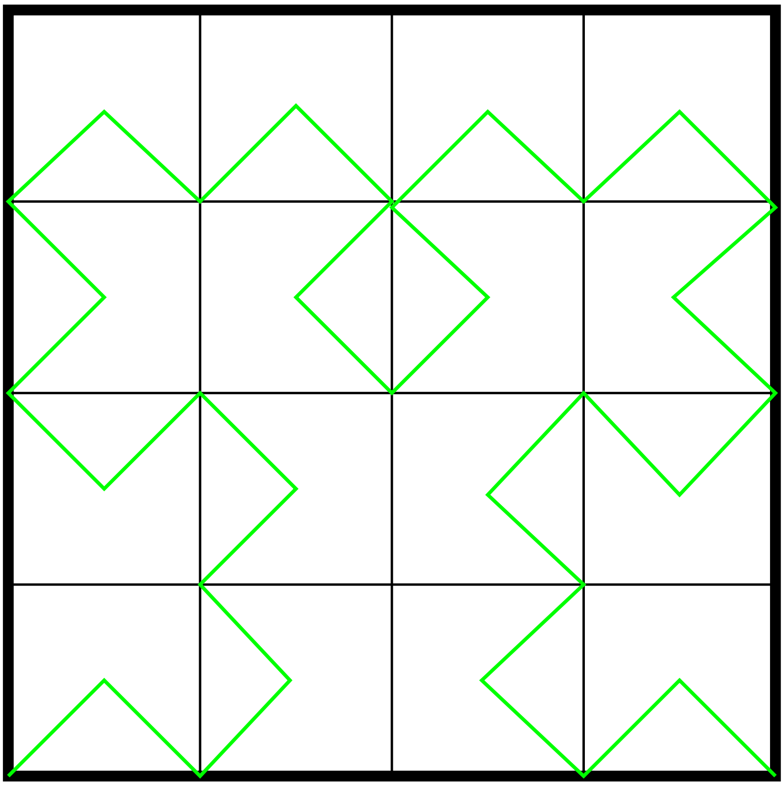
The first such arcs were constructed by Osgood in 1903.

These arcs are not, of course, space-filling curves, but the construction is similar.

First let us recall how we can obtain space-filling curves, using the following pictures.

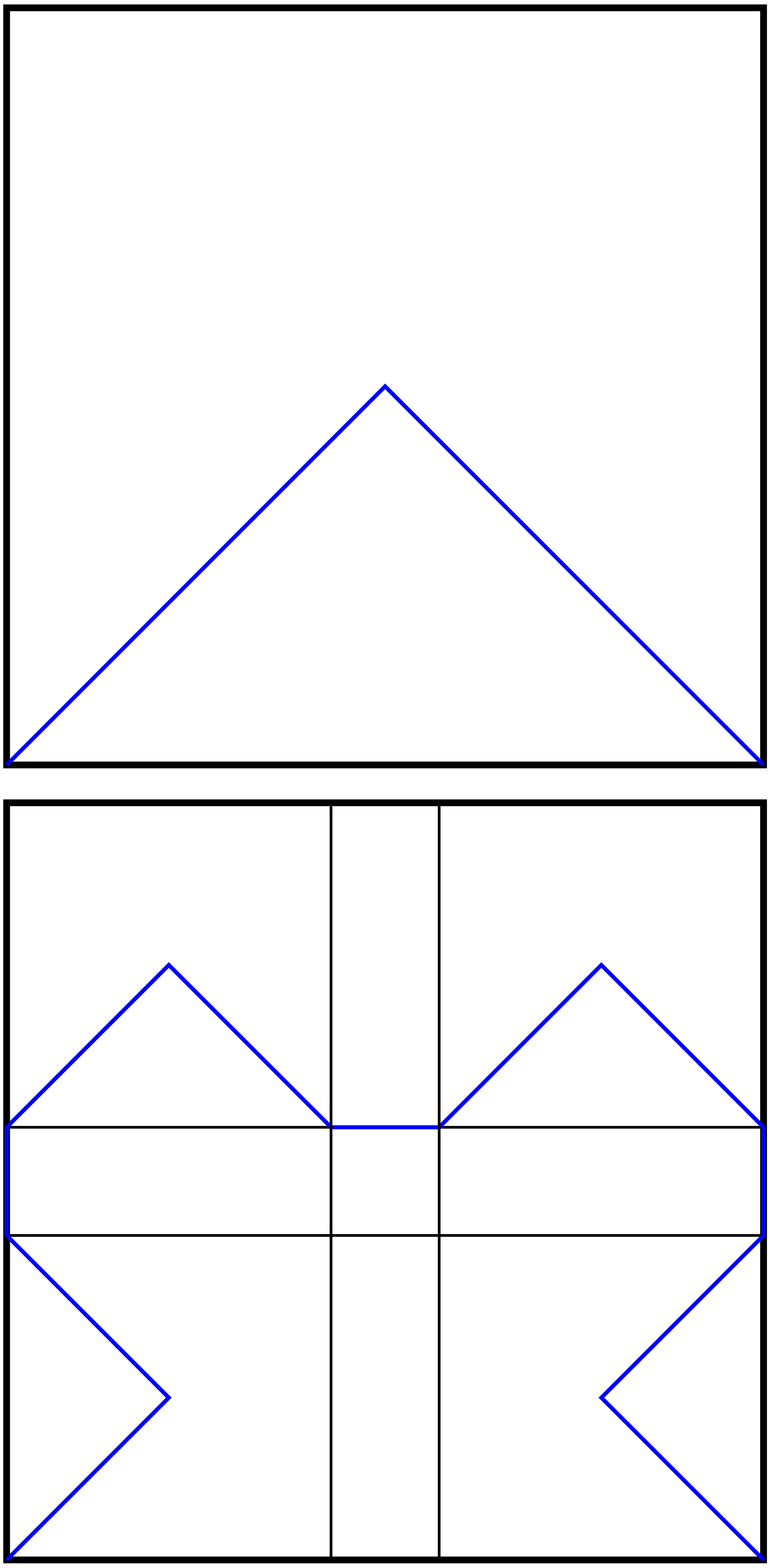


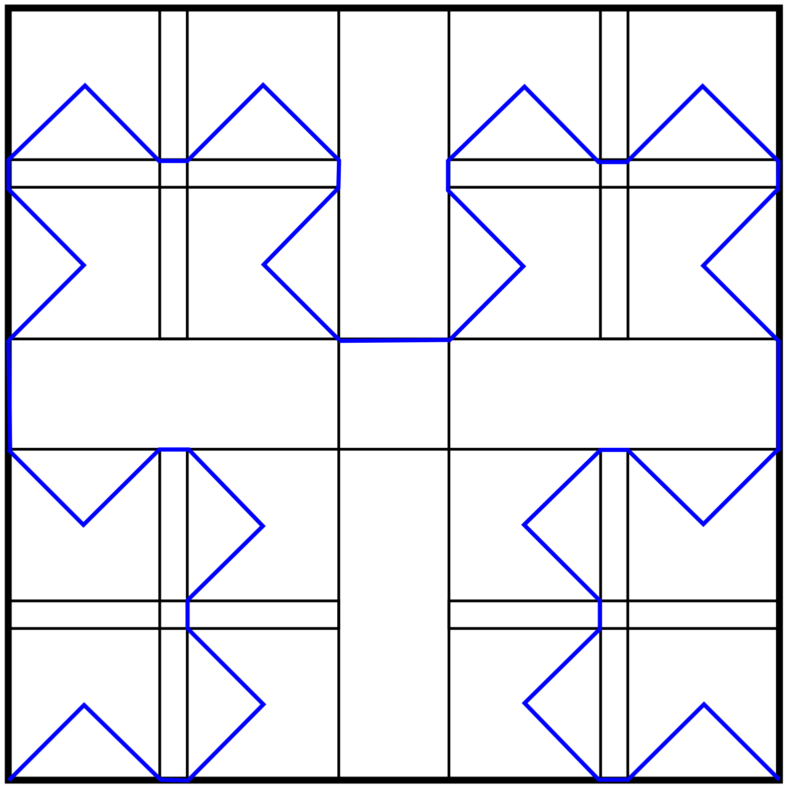




Gap to fill in

A slight modification of this construction produces arcs with positive area.





Gap to fill in

We can now describe some non-trivial uniform algebras on the unit interval.

These examples are not, however, natural on $[0, 1]$

Gap to fill in

5.2 Progress on Gelfand's problem

We now discuss some of the results related to Gelfand's problem.

Proposition 5.2.1 (D. Wilken, 1969) The only strongly regular uniform algebra on $[0, 1]$ is $C[0, 1]$.

This result was strengthened somewhat by Feinstein and Somerset ('Strong regularity for uniform algebras', 1998), but it is an open question whether or not every (natural) regular uniform algebra on $[0, 1]$ is trivial.

Note that Wilken showed in 1965 that every natural uniform algebra on $[0, 1]$ is 'approximately normal'.

Gap to fill in

A rather different attack on the problem may be found in the paper of Dawson and Feinstein (2003).

Proposition 5.2.2 Let A be a natural uniform algebra on $[0,1]$.

Suppose that $\mathrm{Inv}\, A$ is dense in A. Then $A = C[0,1]$.

Here the condition that $\mathrm{Inv}\, A$ is dense in A says that A has **topological stable rank** equal to 1.

Gap to fill in

In the case where A is finitely generated as a Banach algebra, Gelfand's problem comes down to a question about $P(J)$ for polynomially convex arcs J in C^N.

Under some fairly mild conditions on the polynomially convex arc you can see that $\mathrm{Inv}\, P(J)$ must be dense in $P(J)$, and so $P(J) = C(J)$.

However, even in this setting, the solution to Gelfand's problem remains elusive.

Gap to fill in

5.3 Open problems

Here we collect together some of the open problems in this area.

Many of these have already been discussed in this course.

Question 5.3.1 Is there a non-trivial, natural uniform algebra on $[0, 1]$?

Question 5.3.2 Is there a non-trivial, natural, regular uniform algebra on $[0, 1]$?

Question 5.3.3 Is there a non-trivial uniform algebra which has spectral synthesis?

Question 5.3.4 Let X be a compact plane set such that $R(X) \neq C(X)$. Can $R(X)$ be strongly regular? Can $R(X)$ have spectral synthesis?

Question 5.3.5 Let A be a regular Banach function algebra on Φ_A, and let $x \in \Phi_A$.

Suppose that M_x has a b.a.i.

Does it follow that A is strongly regular at x?

(This question is also open for uniform algebras.)

Question 5.3.6 Let A be a Banach function algebra on Φ_A.

Suppose that the only closed ideals in A are the kernels $I(E)$ for closed sets $E \subseteq \Phi_A$.

Does it follow that A is regular, and hence that A has spectral synthesis?