2 Introduction to normed algebras and Banach algebras

2.1 Some problems to think about

Those who have seen much of this introductory material before may wish to think about some of the following problems.

We shall return to these problems at suitable points in this course.

Problem 2.1.1 (Easy using standard theory!) It is standard that the set of all rational functions (quotients of polynomials) with complex coefficients is a field: this is a special case of the “field of fractions” of an integral domain.

Question: Is there an algebra norm on this field (regarded as an algebra over $\mathbb{C}$)?

Problem 2.1.2 (Very hard!) Does there exist a pair of sequences (λ_n), (a_n) of non-zero complex numbers such that

(i) no two of the a_n are equal,

(ii) $\sum_{n=1}^{\infty} |\lambda_n| < \infty$,

(iii) $|a_n| < 2$ for all $n \in \mathbb{N}$, and yet,

(iv) for all $z \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \lambda_n \exp(a_n z) = 0?$$

Gap to fill in

Problem 2.1.3 Denote by $C[0,1]$ the "trivial" uniform algebra of all continuous, complex-valued functions on $[0,1]$.

(i) **(Very hard!)** Give an example of a proper, uniformly closed subalgebra A of $C[0,1]$ such that A contains the constant functions and separates the points of $[0,1]$.

In other words, give an example of a non-trivial uniform algebra on $[0,1]$.

(ii) **(Impossible?)** Is there an example of an algebra A as in (i) with the additional property that the only non-zero, multiplicative linear functionals on A are the evaluations at points of $[0,1]$?

In other words, is there a non-trivial uniform algebra whose character space is $[0,1]$?

(This is a famous open problem of Gelfand.)

2.2 Revision of basic definitions

Banach algebras may be thought of as Banach spaces with multiplication (in a sense made more formal below).

The additional structure provided by the multiplication gives the theory of Banach algebras a rather different flavour from the more general theory of Banach spaces.

Banach algebras may be real or complex. However the theory of complex Banach algebras is richer, and so this is what we will focus on.

Banach algebras may be commutative or non-commutative. We will focus mainly on commutative Banach algebras.

A key example of a commutative Banach algebra is $C_{\mathbb{C}}(X)$, the algebra of continuous, complex-valued functions on a compact, Hausdorff space, with the usual pointwise operations and with the uniform norm.

A typical non-commutative Banach algebra is $B(E)$, the algebra of all bounded linear operators from E to E for some Banach space E (of dimension at least 2!), with the usual vector space structure and operator norm, and with product given by composition of operators.

In particular, for a Hilbert space H, $B(H)$ and its subalgebras are of major interest.

Just as the theory of Banach algebras should not be regarded as part of the theory of Banach spaces, the theory of $B(H)$ and its subalgebras has its own flavour and a vast literature, including the quite distinct study of C^*-algebras and von Neumann algebras.

Definition 2.2.1 A **complex algebra** is a complex vector space A which is a ring with respect to an associative multiplication which is also a bilinear map, i.e., the distributive laws hold and, for all $\alpha \in \mathbb{C}$ and a and b in A, we have

$$(\alpha a)b = a(\alpha b) = \alpha(ab)\,.$$

The complex algebra A **has an identity** if there exists an element $e \neq 0 \in A$ such that, for all $a \in A$, we have $ea = ae = a$.

Real algebras are defined similarly.

Note that, if A has an identity e, then this identity is unique, so we may call e **the identity** of A.

We will often denote the identity by 1 rather than e, assuming that the context ensures that there is no ambiguity.

The assumption above that $e \neq 0$ means that we do not count 0 as an identity in the trivial algebra $\{0\}$.

From now on, all algebras will be assumed to be complex algebras, so we will be defining **complex** normed algebras and Banach algebras.

Definition 2.2.2 A **(complex) normed algebra** is a pair $(A, \|\cdot\|)$ where A is a complex algebra and $\|\cdot\|$ is a norm on A which is **sub-multiplicative**, i.e., for all a and b in A, we have

$$\|ab\| \leq \|a\|\|b\| .$$

A normed algebra A is **unital** if it has an identity 1 **and** $\|1\| = 1$.

A **Banach algebra** (or **complete normed algebra**) is a normed algebra which is complete as a normed space.

Notes.

- In our usual way, we will often call A itself a normed algebra, if there is no ambiguity in the norm used on A.

- We will mostly be interested in commutative, unital Banach algebras.

- The condition that the norm on A be sub-multiplicative is only slightly stronger than the requirement that multiplication be jointly continuous (or equivalently, that multiplication be a 'bounded bilinear map') from $A \times A \to A$.

  In fact (**easy exercise**) if we have $\|ab\| \leq C\|a\|\|b\|$, for some constant $C > 1$, then we can easily find an equivalent norm on A which is actually sub-multiplicative.

- If a normed algebra $(A, \|\cdot\|)$ has an identity 1 such that $\|1\| \neq 1$, then we may again define another norm $|||\cdot|||$ on A as follows:

Left mult.

closed unit ball in A.

$$\|T_a\|_{op} = \quad |||a||| = \sup\{\|ab\| : b \in \bar{B}_A(0,1)\}.$$

$B_A(x,r)$ = open ball centred on x radius r

$\bar{B}_A(x,r)$ = closed ball.

As another **exercise**, you may check that $|||\cdot|||$ is equivalent to $\|\cdot\|$, that $|||\cdot|||$ is sub-multiplicative and that $|||1||| = 1$.
So the assumption that $\|1\| = 1$ is a convenience, rather than a topological restriction.

- Every subalgebra of a normed algebra is a normed algebra, and every closed subalgebra of a Banach algebra is a Banach algebra.

Examples.

operator norm

(1) Let E be a complex Banach space of dimension > 1. Then $(B(E), \|\cdot\|_{\mathrm{op}})$ is a non-commutative, unital Banach algebra, where the product on $B(E)$ is composition of operators.

(2) Let X be a non-empty, compact, Hausdorff topological space. Then $C_{\mathbb{C}}(X)$ is a commutative, unital Banach algebra with pointwise operations and the uniform norm $|\cdot|_X$: recall that

$$\|f\|_\infty = |f|_X = \sup\{|f(x)| : x \in X\}.$$

As we are focussing on complex algebras, **from now on we will denote** $C_{\mathbb{C}}(X)$ **by** $C(X)$.

(3) With notation as in (2), every (uniformly) closed subalgebra of $C(X)$ is also a commutative Banach algebra with respect to the uniform norm $|\cdot|_X$.

A **uniform algebra on** X is a uniformly closed subalgebra A of $C(X)$ which contains all the constant functions and which **separates the points of** X, i.e., whenever x and y are in X with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$.

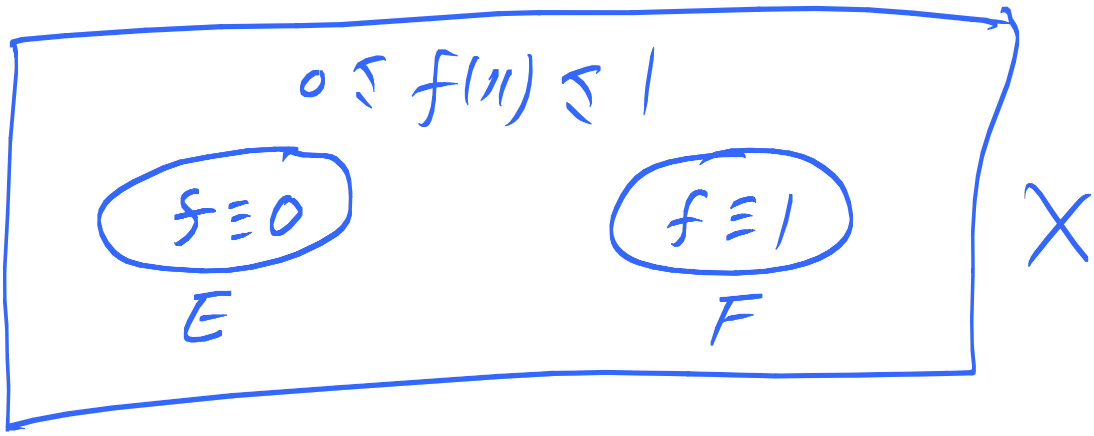
Every uniform algebra on X is, of course, a commutative, unital Banach algebra, with respect to the uniform norm.

X compact

Also, by Urysohn's Lemma, $C(X)$ itself **does** separate the points of X, and so $C(X)$ is a uniform algebra on X.

E, F closed $\subseteq X$, $E \cap F = \emptyset$

Gap to fill in



$\exists\, f \in C(X), \quad f(E) \subseteq \{0\},\ f(F) \subseteq \{1\}$

and $\quad 0 \leq f(x) \leq 1 \quad$ all x in X.

<u>N.b.</u> Compact Hausdorff spaces are "normal" topological spaces.

So are all metric spaces,

but (in spite of what I said in the audio!) <u>Locally</u> compact Hausdorff spaces need not be normal.

[Locally compact, Hausdorff spaces <u>are</u> <u>regular</u> topological spaces, though.]

(4) In particular, let X be a non-empty, compact subset of $\mathbb{C}$. (or $\mathbb{C}^N$ for $N \in \mathbb{N}$)

Consider the following subalgebras of $C(X)$:

$A(X)$ is the set of those functions in $C(X)$ which are analytic (holomorphic) on the interior of X;

$P_0(X)$ is the set of restrictions to X of polynomial functions with complex coefficients;

$R_0(X)$ is the set of restrictions to X of rational functions with complex coefficients whose poles (if any) lie off X (so
$R_0(X) = \{p/q : p, q \in P_0(X), 0 \notin q(X)\}$).

It is easy to see that these subalgebras contain the constant functions and separate the points of X.

Indeed the polynomial function Z, also called the **co-ordinate functional**, defined by $Z(\lambda) = \lambda$ $(\lambda \in \mathbb{C})$, clearly separates the points of X by itself, and is in all of these algebras.

The algebra $A(X)$ is closed in $C(X)$, because uniform limits of analytic functions are analytic (see books for details), and so $A(X)$ is a uniform algebra on X.

The algebras $P_0(X)$ and $R_0(X)$ are usually not closed in $C(X)$ (**exercise:** investigate this).

We obtain uniform algebras on X by taking their (uniform) closures: $P(X)$ is the closure of $P_0(X)$ and $R(X)$ is the closure of $R_0(X)$.

The functions in $P(X)$ are those which may be **uniformly approximated** on X by polynomials, and the functions in $R(X)$ are those which may be **uniformly approximated** on X by rational functions with poles off X.

We have $P(X) \subseteq R(X) \subseteq A(X) \subseteq C(X)$.

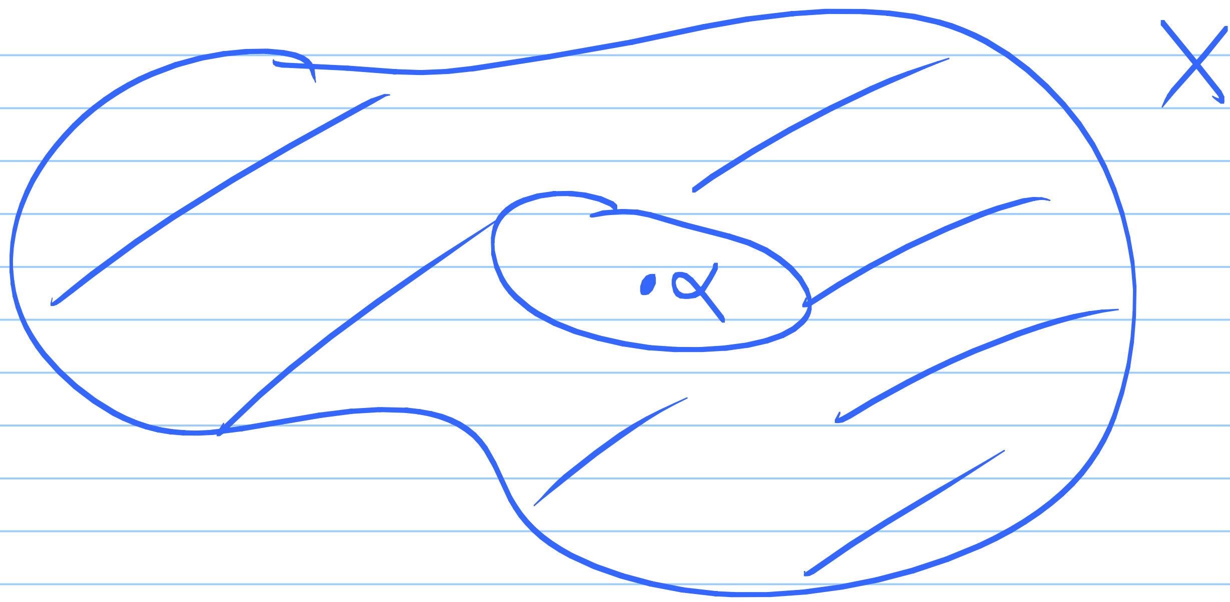
Gap to fill in

$\underline{X \subseteq \mathbb{C}}$

$P(X) = R(X) \iff X$ has "no holes".

$\mathbb{C} \setminus X$ connected

$\mathbb{C} \setminus X$ has two bounded components.



$$z - \alpha 1 \in P(X),$$

but is not invertible in $P(X)$.

$$\frac{1}{z - \alpha 1} \in R(X) \setminus P(X).$$

<u>Converse</u> Runge's Theorem:

if $\mathbb{C} \setminus X$ is connected, then

$P(X) = R(X)$ (and more).

[In fact $P(X) = A(X)$ here, by Mergelyan's Theorem.]

(5) The Banach space $C^1[0,1]$ of once continuously differentiable complex-valued functions on $[0,1]$ is a subalgebra of $C[0,1]$.

It is not uniformly closed, and so it is not a uniform algebra on $[0,1]$.

However, it is a Banach algebra when given its own norm, $\|f\| = |f|_X + |f'|_X$ (where $X = [0,1]$).

This is a typical example of a **Banach function algebra**.

Warning! In the literature, some authors call uniform algebras **function algebras**.

Gap to fill in

(6) Every complex normed space E can be made into a (non-unital) normed algebra by defining the trivial multiplication $xy = 0$ $(x, y \in E)$.

Thus every complex Banach space is also a Banach algebra.

(7) The following standard construction can be used to 'unitize' Banach algebras.

Let $(A, \|\cdot\|_A)$ be a complex Banach algebra **without** an identity.

We may form a Banach algebra $A^\#$, called the **standard unitization** of A, as follows.

As a vector space, $A^\# = A \oplus \mathbb{C}$.

This becomes a Banach space when given the norm

$$\|(a,\alpha)\| = \|a\|_A + |\alpha| \quad (a \in A, \alpha \in \mathbb{C})\,.$$

We can then make $A^\#$ into a unital Banach algebra using the following multiplication: for a and b in A and α and β in $\mathbb{C}$, we define

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta)\,.$$

(a,α) "is" $a + \alpha 1$

Exercise. Check the details of these claims.

(What is the identity element of $A^\#$?)

Show also that $A^\#$ is commutative if and only if A is commutative.

We can combine this construction with (6) to give examples of commutative, unital Banach algebras which have many non-zero elements whose squares are 0.

This never happens, of course, for our algebras of functions.

2.3 Characters and the character space for commutative Banach algebras

In this section we will focus on commutative algebras. However, many of the definitions and results are valid (with some minor modifications) in the non-commutative setting too: see books for details.

Definition 2.3.1 Let A be a commutative algebra with identity 1.

An element $a \in A$ is **invertible** if there exists $b \in A$ with $ab = 1$.

In this case the element b is unique: b is then called the **inverse** of a, and is denoted by a^{-1}.

The set of invertible elements of A is denoted by $\operatorname{Inv} A$.

Notes.

- For non-commutative algebras, we would insist that both $ab = 1$ and $ba = 1$.
- For invertible elements a, it is clear that a^{-1} is invertible and $(a^{-1})^{-1} = a$.

- With multiplication as in A, $\operatorname{Inv} A$ is a group with identity 1, and the map $a \mapsto a^{-1}$ is a bijection from $\operatorname{Inv} A$ to itself.

- Let X be a non-empty, compact, Hausdorff topological space. Then

$$\operatorname{Inv} C(X) = \{f \in C(X) : 0 \notin f(X)\}\,.$$

  If X is a non-empty, compact subset of $\mathbb{C}$ then the same is true for $R(X)$ and $A(X)$, but not for $P(X)$ unless X has 'no holes' (see books for more details on this).

Gap to fill in

$R(X)$, $A(X)$ easy,

$P(X)$ as above

$z - \alpha 1$ for α in some bounded component of $\mathbb{C} \setminus X$.

Theorem 2.3.2 Let A be a commutative, unital Banach algebra, and let $x \in A$ with $\|x\| < 1$. Then $1 - x$ is invertible, and

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots .$$

Thus whenever $a \in A$ with $\|a - 1\| < 1$, we have $a \in \operatorname{Inv} A$.

In other words, the open ball in A centred on the identity element and with radius 1 is a subset of $\operatorname{Inv} A$.

$B_A(1, 1) \subseteq \mathrm{Inv}(A)$

We now investigate characters and the character space of commutative, unital Banach algebras.

Definition 2.3.3 Let A be a commutative algebra.

A **character** on A is a **non-zero, multiplicative linear functional** on A, i.e. a non-zero linear functional $\phi : A \to \mathbb{C}$ satisfying $\phi(ab) = \phi(a)\phi(b)$ $(a, b \in A)$.

The set of all characters on A is called the **character space** of A, and is denoted by Φ_A.

Notes.

- Suppose that A has an identity, 1. Then it is elementary to show that $\phi(1) = 1$ for all $\phi \in \Phi_A$.

  It is also easy to show, in this case, that Φ_A is a closed subset of the product space $\mathbb{C}^A$. $\prod_{a \in A} \mathbb{C}$

- If A is an algebra of complex-valued functions on a non-empty set X (with pointwise operations), then, for every $x \in X$, there is an **evaluation character** $\varepsilon_x = \hat{x}$ at x, denoted by $\hat{x}$, defined by $\hat{x}(f) = f(x)$ $(f \in A)$.

  In general there may also be many other characters on A.

  However, it turns out that in the case of $C(X)$ (for compact, Hausdorff X) there are no others.

For non-empty compact subsets X of $\mathbb{C}$ (also known as **compact plane sets**), the same is true for $R(X)$ and for $A(X)$, but not for $P(X)$ unless X has 'no holes'.

See books for details: $A(X)$ is rather hard!

Gap to fill in

In $\mathbb{C}^N$, result for $R(X)$ is false unless X is "rationally convex".

The following is the most basic of the 'automatic continuity' results concerning Banach algebras.

Theorem 2.3.4 Let A be a commutative, unital Banach algebra.

Then every character ϕ on A is continuous, with $\|\phi\| = 1$.

Using this and the Banach-Alaoglu Theorem, we obtain the following important corollary.

Corollary 2.3.5 Let A be a commutative, unital Banach algebra.

Then Φ_A is a weak-* compact subset of A^*.

The relative (i.e. subspace) weak-* topology on Φ_A is called the **Gelfand topology**.

The Gelfand topology is the weakest topology on Φ_A such that, for all $a \in A$, the map $\phi \mapsto \phi(a)$ is continuous.

We may restate the above corollary as follows:

Φ_A is a compact, Hausdorff topological space with respect to the Gelfand topology.

By default, we will always use the Gelfand topology on Φ_A. (But see later for hull-kernel topology)

In fact, every commutative, unital Banach algebra has at least one character: we will return to this later.

We conclude this section by recalling the definition of the **Gelfand transform**.

Definition 2.3.6 Let A be a commutative, unital Banach algebra. Then the **Gelfand transform** is the map from A to $C(\Phi_A)$ defined by $a \mapsto \hat{a}$, where $\hat{a}(\phi) = \phi(a)$ $(a \in A, \phi \in \Phi_A)$.

The **Gelfand transform of** A is the set $\widehat{A} = \{\hat{a} : a \in A\}$.

2.4 Semisimple, commutative, unital Banach algebras

You probably already know various definitions of the term semisimple.

We give our definition in terms of characters.

See books for the equivalence of this and the usual algebraic definition, in the setting of commutative, unital Banach algebras.

Definition 2.4.1 Let A be a commutative, unital Banach algebra. Then A is semisimple if

$$\bigcap_{\phi \in \Phi_A} \ker \phi = \{0\},$$

i.e., for every non-zero $a \in A$, there exists a character ϕ on A with $\phi(a) \neq 0$.

so $\hat{a}(\phi) \neq 0$

$\hat{a}$ is not the zero function.

Notes.

- Every unital Banach algebra of functions on a set X is semisimple.

  This may be seen immediately by considering just the evaluation characters at points of X.

- Conversely, every semisimple, commutative, unital Banach algebra A is isomorphic (as an algebra) to a subalgebra of $C(\Phi_A)$.

  Indeed A is semisimple if and only if the Gelfand transform is injective, in which case A is isomorphic to its Gelfand transform $\widehat{A}$.

We have implicitly assumed above the obvious notions of **algebra homomorphism** (a multiplicative linear map) and **algebra isomorphism** (a bijective algebra homomorphism).

We also need the notion of a **unital** algebra homomorphism.

Definition 2.4.2 Let A and B be commutative, unital Banach algebras, with identities 1_A and 1_B respectively.

A **unital** algebra homomorphism from A to B is an algebra homomorphism $T : A \to B$ such that $T(1_A) = 1_B$.

The following remarkable Automatic Continuity results are true concerning semisimple, commutative, unital Banach algebras.

The first result concerns the automatic continuity of homomorphisms.

Theorem 2.4.3 Let A and B be commutative, unital Banach algebras, and suppose that B is semisimple. Then every unital algebra homomorphism from A to B is automatically continuous.

not necessarily onto.

The next result (a corollary) is a **uniqueness of norm** result.

Corollary 2.4.4 Let $(A, \|\cdot\|)$ be a semisimple, commutative, unital Banach algebra.

Suppose that $\|\cdot\|'$ is another norm on A such that $(A, \|\cdot\|')$ is a commutative unital Banach algebra. Then the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Gap to fill in

Easy exercise: deduce 2.4.4 from 2.4.3.

Proof of Thm 2.4.3

We just use automatic continuity of characters + closed graph thm.

Let A, B be as in statement.

Let $T : A \to B$ be a

unital algebra homomorphism.

Let $(a_n) \subseteq A$ with $a_n \to 0$ as $n \to \infty$, and suppose $Ta_n \to b \in B$ as $n \to \infty$.

We show that $b = 0$.

(Note $T(1) = 1$.)

Let $\phi \in \Phi_B$.

Then $(\phi \circ T) \in \Phi_A$.

(Note $\phi(T(1)) = 1$).

Thus $\phi \circ T$ is continuous.

Since $a_n \to 0$, $\phi(T(a_n)) \to 0$ as $n \to \infty$.

But $T(a_n) \to b$ as $n \to \infty$.

Since ϕ is cts, $\phi(T(a_n)) \to \phi(b)$ as $n \to \infty$. Thus $\phi(b) = 0$.

This holds for all $\phi \in \Phi_B$.

Since B is semisimple, $b = 0$. The result follows. $\square$

Non-commutative version

Johnson's theorem, Johnson's uniqueness of norm

Theorem. Let A, B be unital Banach algebras, with B semisimple. Let $T: A \to B$ be a surjective (unital) algebra homomorphism. Then T is automatically cts.

What if $T(A) \neq B$?

$\boxed{\text{Is it enough to assume } T(A) \text{ is dense in } B?}$

Big open problem!

2.5 Resolvent and spectrum

We begin this section with some further facts concerning the invertible group $\mathrm{Inv}\, A$ of a commutative, unital Banach algebra A.

For convenience, we will use the abbreviation CBA for commutative Banach algebra.

Theorem 2.5.1 Let A be a unital CBA. Then the following facts hold:

(a) $\mathrm{Inv}\, A$ is open in A;

(b) the map $a \mapsto a^{-1}$ is a homeomorphism from $\mathrm{Inv}\, A$ to itself.

We now define the **resolvent set** and the **spectrum** for an element of a commutative algebra with identity.

Definition 2.5.2 Let A be a commutative algebra with identity and let $x \in A$.

Then the **spectrum of x in A**, $\sigma_A(x)$ (or $\sigma(x)$ if the algebra under consideration is unambiguous) is defined by

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \notin \mathrm{Inv}\, A\}.$$

The **resolvent set of x in A**, $\rho_A(x)$, is the complement of the spectrum, i.e.,

$$\rho_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \in \operatorname{Inv} A\}\,.$$

Notes. We are mainly interested in the case of CBA's. So, let A be a unital CBA.

- Since $\operatorname{Inv} A$ is open in A, it follows easily that the resolvent set is open in $\mathbb{C}$, and hence that the spectrum is closed in $\mathbb{C}$.
- We have $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\,\}$. Thus the spectrum is always a compact subset of $\mathbb{C}$.

The next result is a consequence of Liouville's Theorem (complex analysis).

See books for the elegant details.

Proposition 2.5.3 Let A be a commutative, unital normed algebra. Then, for all $x \in A$, $\sigma(x) \neq \emptyset$.

The next result shows that $\mathbb{C}$ itself is the only complex normed algebra which is a field.

Theorem 2.5.4 (Gelfand-Mazur) Let A be a commutative, unital normed algebra.

Suppose that $\operatorname{Inv} A = A \setminus \{0\}$. Then $A = \operatorname{lin}\{1\}$, and A is isometrically isomorphic to $\mathbb{C}$.

Thus none of the many non-trivial extension fields of $\mathbb{C}$ can be given a (complex) algebra norm.

Gap to fill in

Proof. Given A as above,
Let $x \in A$. Then $\sigma_A(x) \neq \emptyset$.
Let $\lambda \in \sigma_A(x)$.

Then $x - \lambda 1$ is not invertible.
So $x - \lambda 1 = 0$. $\square$

NOTE:

Don't need commutative.
$\mathbb{C}$ is the only normed division algebra over $\mathbb{C}$ ...
note that the quaternions are NOT a complex algebra!

We conclude this section with the definition and formula for the spectral radius.

Definition 2.5.5 Let A be a commutative, unital Banach algebra, and let $x \in A$.

We define the **spectral radius** of x (in A), $\nu_A(x)$, by

$$\nu_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}\,.$$

The following result is the famous **spectral radius formula**.

Theorem 2.5.6 Let A be a commutative, unital Banach algebra, and let $x \in A$. Then

$$\nu_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n}\,.$$

Gap to fill in

2.6 Maximal ideals and characters

When working with algebras rather than just rings, we insist that an **ideal** in the algebra must be a linear subspace, in addition to being an ideal in the ring theory sense.

Fortunately, this makes no difference in the case where the algebra has an identity.

Definition 2.6.1 Let A be a commutative algebra with identity. Then a **proper ideal** in A is an ideal I in A such that $I \neq A$.

A **maximal ideal** in A is a maximal proper ideal in A (with respect to set inclusion).

An easy Zorn's Lemma argument shows that every proper ideal I in A is contained in at least one maximal ideal in A.

We conclude this introductory chapter with a standard result which connects up the concepts discussed so far.

Theorem 2.6.2 Let A be a unital CBA. Then the following hold.

(a) Every maximal ideal in A is closed.

(b) For every character $\phi \in \Phi_A$, $\ker \phi$ is a maximal ideal in A.

(c) Conversely, every maximal ideal in A is the kernel of a unique character on A. In particular, every maximal ideal has codimension 1 in A.

(d) For every $x \in A$, we have

$$\sigma_A(x) = \{\phi(x) : \phi \in \Phi_A\}.$$

(e) The character space Φ_A is non-empty, and, for all $x \in A$, the spectrum of x is equal to the image of the Gelfand transform of x, i.e.,

$$\sigma_A(x) = \hat{x}(\Phi_A).$$

Gap to fill in

Gap to fill in