Lecture 3: LSZ reduction formula. Path integral in QM

- Interacting theory scattering amplitudes.
- Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.
- Path integral formulation of quantum mechanics.
One simple formula

The creation-annihilation operators can be expressed in terms of the field as follows:

\[ a_p = i \int d^3x \ e^{-i k x} \rightarrow \partial_0 \phi(x, t), \]

\[ a_p^\dagger = -i \int d^3x \ e^{i k x} \leftarrow \partial_0 \phi(x, t). \]
One identity

Construct a creation-annihilation operator pair for a “wave-packet”:

\[ a_f^\dagger = \int d^3k \ f(k) a_k^\dagger. \]

Then a relation between the creation-annihilation operators in the “future” and in the “past” is:

\[ a_f(+\infty) = a_f(-\infty) + i \int d^3k \ f(k) \int d^4x \ e^{-ikx} \left[ -\partial_\mu \partial^\mu + m^2 \right] \phi(x, t), \]

\[ a_f^\dagger(-\infty) = a_f^\dagger(+\infty) + i \int d^3k \ f(k) \int d^4x \ e^{ikx} \left[ -\partial_\mu \partial^\mu + m^2 \right] \phi(x, t), \]
The scattering amplitude

Introduce the “scattering” amplitude for wave packets:

\[ \langle f | i \rangle = \langle 0 | a_{f_1}(+\infty) \ldots a_{f_n}(+\infty) a_{g_1}'(-\infty) \ldots a_{g_n}'(-\infty) | 0 \rangle = \langle 0 | Ta_{f_1}(+\infty) \ldots a_{f_n}(+\infty) a_{g_1}'(-\infty) \ldots a_{g_n}'(-\infty) | 0 \rangle. \]
LSZ reduction formula

Use the identities on the previous page, remove the form-factors $f, g$ to get:

$$\langle f | i \rangle = i^{n+n'} \int d^4 x_1 e^{i k_1 x_1} \left[ -\partial_\mu \partial^\mu + m^2 \right] \ldots \int d^4 x_n e^{i k_n x_n} \left[ -\partial_\mu \partial^\mu + m^2 \right]$$

$$\int d^4 x_1 e^{-i k'_1 x'_1} \left[ -\partial_\mu \partial^\mu + m^2 \right] \ldots \int d^4 x_n e^{-i k'_n x'_n} \left[ -\partial_\mu \partial^\mu + m^2 \right]$$

$$\langle 0 | T \phi(x_1) \ldots \phi(x_n) \phi(x'_1) \phi(x'_n) | 0 \rangle.$$ 

Thus, everything we need to know are time-ordered correlators!
Path integral formulation of quantum mechanics

Quantum mechanics can be reformulated very simply by saying that the quantum amplitude of any trajectory is given by $A[q(t)] \sim e^{iS[q(t)]/\hbar}$. Amplitudes of all possible trajectories add up, to give the full amplitude to go from one place to another.

This is summarized in the notion of path integral:

$$A[q''(t'')|q'(t')] = \int_{q[t'']=q''}^{q'[t']=q'} Dq[t] \ e^{iS[q[t]]/\hbar}.$$ 

Probability is equal by the absolute value squared of the amplitude: $P = |A|^2$. Put $\hbar = 1$ from now.
“Derivation” of the path integral

\[ H\langle q''t''|q't'\rangle_H = S\langle q''|e^{-iH(t''-t')}|q'\rangle S. \]

Split the time interval into \( \delta t = (t'' - t')/(N + 1) \) and insert a complete set of states at each step to find, as \( N \to \infty \):

\[ H\langle q''t''|q't'\rangle_H = \int \mathcal{D}p \mathcal{D}q \exp i \int_{t'}^{t''} dt [p(t)\dot{q}(t) - H(p(t), q(t))]. \]
Correlation functions from the path integral

\[ H\langle q''t''|\hat{q}(t_1)|q't'\rangle_H = s\langle q''|e^{-iH(t''-t_1)}\hat{q}(t_1)e^{-iH(t_1-t')}|q'\rangle_S = \int Dp\, Dq\, q(t_1) e^{iS}. \]

More generally:

\[ H\langle q''t''|T\hat{q}(t_1)\hat{q}(t_2)|q't'\rangle_H = \int Dp\, Dq\, q(t_1)q(t_2) e^{iS}. \]

 Computes the \textit{time-ordered correlation functions}. 
Vacuum-vacuum amplitudes

The trick of making the time “a little imaginary”:

\[
\langle 0|0 \rangle = \int Dp\, Dq\, e^{i \int_{-\infty}^{+\infty} dt [p\dot{q} - (1-i\epsilon)H]}.\]
Generating functional

All correlation functions can be put together in the generating functional:

$$\langle 0|0 \rangle_f = \int \mathcal{D}p \, \mathcal{D}q \, e^{i \int_{-\infty}^{+\infty} dt [p\dot{q}-(1-i\epsilon)H+qf]}.$$ 

Vacuum-vacuum amplitude in the presence of a source $f$. The correlation functions by differentiating:

$$\langle 0|T\hat{q}(t_1)\hat{q}(t_2)|0 \rangle = \left( \frac{\delta}{i\delta f(t_1)} \frac{\delta}{i\delta f(t_2)} \langle 0|0 \rangle_f \right)_{f=0}.$$