Lecture 12: Power counting and renormalization

Which theories should be expected to be renormalizable and which should not?

- Mass dimensions.
- Superficial degree of divergence of a diagram.
- Criterion for non-renormalizability.
- Degree of divergence in terms of the mass dimensions of the couplings.
- List of renormalizable theories in 4 dimensions.

General Lagrangian

Consider the most general interacting scalar field theory:

$$\mathcal{L} = -\frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - \frac{1}{2} Z_m m^2 \phi^2 - \sum_{n=3}^{\infty} \frac{1}{n!} Z_n g_n \phi^n .$$

Which of these theories are renormalizable (in a given dimension)?

Mass dimensions of fields and couplings

We have already discussed the concept of the mass dimension.

Mass dimension of the field $[\phi] = (d-2)/2$.

Mass dimension of the coupling constant: $[g_n] = d - n(d-2)/2$.

Superficial degree of divergence

Consider a Feynman diagram.

(Superficial degree of divergence D) $=$ The power of momenta in the numerator $-$ the power of momenta in the denominator

Diagram diverges if its degree of divergence is $D \geq 0$. May also diverge even if $D < 0$ because of the divergent subgraphs.

A formula for the degree of divergence:

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A formula for the degree of divergence:

$$D = dL - 2I.$$

Here L is the number of (independent) loops in the diagram (momentum integrations), and I is the number of internal edges in it.

Renormalizability

D is a quantitative indicator of how the high energy physics affects this particular particle process.

Diagrams with the same (or smaller) D can be combined together in the renormalization procedure.

A theory is renormalizable if only a finite number of positive D's is possible in it. Then there is only a finite number of "blocks" of diagrams that have to be considered and taken care of by the multiplicative renormalization.

If the set of possible positive D's arising is infinite, then the theory is non-renormalizable. An infinite number of multiplicative renormalizations is necessary. There is an infinite number of physical parameters that have to be fixed by experiments. No (or very little) predictive power.

Degree of divergence in terms of mass dimensions

Consider (LSZ truncated) diagrams with open ends. The mass dimension of such a diagram is equal to the dimension of the corresponding coupling constant: $[diagram] = [g_E]$.

On the other hand, the mass dimension can be computed as: $[diagram] = dL - 2I + \sum_n V_n[g_n]$, where V_n is the number of vertices of valency n in the diagram.

Therefore,

$$\boxed{D = [g_E] - \sum_{n=3}^{\infty} V_n[g_n]}.$$

Criterion for non-renormalizability

If any of the coupling constants has a negative mass dimension, can get large and large positive degree of divergence by adding vertices of this type.

$$\text{Non-renormalizable} \quad \Leftrightarrow \quad [g_n] < 0$$

If all $[g_n]$ are positive or zero - necessary, but not sufficient condition for the renormalizability.

List of renormalizable theories in 4 dimensions

- Spin 0,1/2 field theories that are renormalizable by power counting are renormalizable.
- Spin 1 fields are renormalizable only if are associated with a gauge symmetry.

Nature uses only spin 1/2 and 1 fields. The standard model of elementary particles also predicts a spin 0 field (Higgs) but it has not been found as of yet.