Lecture 1: Introduction to QFT and Second Quantization

- General remarks about quantum field theory.
- What is quantum field theory about?
- Why relativity plus QM imply an unfixed number of particles?
- Creation-annihilation operators.
- Second quantization.

General remarks

- Quantum Field Theory as the theory of "everything": all other physics is derivable, except gravity.
- The pinnacle of human thought. The distillation of basic notions from the very beginning of the physics.
- May seem hard but simple and beautiful once understood.

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- The number of particles is unfixed.
- QFT can describe relativistic as well as non-relativistic systems.
- QFT's technical and conceptual difficulties stem from it having to describe processes involving an unfixed number of particles.

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- Relativity says that Mass=Energy.
- Quantum mechanics makes any energy available for a short time: $\Delta E \cdot \Delta t \sim \hbar$

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Solution: Eigenvalues are positive integers n. The action of $a, a^\dagger$ on normalized eigenstates is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle,$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

In particular, there is the "vacuum" state $|0\rangle : a|0\rangle = 0$.

Interpretation: The number n as the number of "quanta" in the state.

Second Quantization

- Consider an arbitrary quantum or classical system, whose states are denoted by $|i\rangle$. E.g. can be eigenstates of the Hamiltonian:

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- Want to alow any number of "quanta" or "particles" in each state $|i\rangle$. A general state is specified by its *occupation numbers* $n_1, n_2, \ldots$:

$$|n_1, n_2, \ldots\rangle$$

Introduce creation-annihilation operators, one copy for each i:

$$[a_i, a_i^\dagger] = 1,$$

$$a_i|n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i}|n_1, n_2, \ldots, n_i - 1, \ldots\rangle,$$
$$a_i^\dagger|n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}|n_1, n_2, \ldots, n_i + 1, \ldots\rangle.$$

The operators $a_i, a_i^\dagger$ create-annihilate particles in state i.

The general state can be built from the vacuum:

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \left[\prod_i \frac{(a_i^\dagger)^{n_i}}{(n_i!)^{1/2}}\right]|0\rangle.$$

Here $|0\rangle : a_i|0\rangle = 0, \forall i$.

Multi-particle states: more convenient notation

$$|i_1, \ldots, i_k\rangle = a^\dagger_{i_1} \ldots a^\dagger_{i_k} |0\rangle$$

If all $i_1, \ldots, i_k$ are all different - correctly normalized. Otherwise norm greater than one.

Relation between the "occupation number" and the "multi-particle" notations:

$$|\underbrace{1, \ldots, 1}_{n_1} \underbrace{2, \ldots, 2}_{n_2}, \ldots\rangle = (a^\dagger_1)^{n_1} (a^\dagger_2)^{n_2} \ldots |0\rangle = \sqrt{n_1! n_2! \ldots} |n_1, n_2, \ldots\rangle$$

The action of creation-annihilation operators on multi-particle states:

$$a_i^\dagger |i_1, \ldots, i_k\rangle = |i, i_1, \ldots, i_k\rangle,$$

$$a_i |i_1, \ldots, i_k\rangle = \sum_{l=1}^{k} \delta_{i i_l} |i_1, \ldots, (\text{no} \;\; i_l), \ldots, i_k\rangle = \sum_{l=1}^{k} \langle i|i_l\rangle |i_1, \ldots, (\text{no} \;\; i_l), \ldots, i_k\rangle.$$

Fock space: Can form a Hilbert space by considering arbitrary linear combinations of multi-particle states. States with different number of particles are orthogonal.

Multi-particle operators: one particle

Problem: Have an operator $A^{(1)}$ that "knows" how to act on 1-particle states. How to extend it on multi-particle states?

Since $|i_1, \ldots, i_k\rangle = |i_1\rangle \otimes \ldots \otimes |i_k\rangle$, then natural to define:

$$\hat{A}|i_1, \ldots, i_k\rangle = A^{(1)}|i_1\rangle \otimes \ldots \otimes |i_k\rangle + \ldots |i_1\rangle \otimes \ldots \otimes A^{(1)}|i_k\rangle.$$

If each state - eigenstate of $A^{(1)}$, then

$$\hat{A}|i_1, \ldots, i_k\rangle = (a_1 + \ldots + a_k)|i_1, \ldots, i_k\rangle.$$

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$$\hat{A} = \sum_{ij} A_{ij} a_i^\dagger a_j.$$

2-Particle operators

$$V^{(2)} = \sum_{ij,kl} V_{ij,kl} |ij\rangle\langle kl|.$$

Want to have an extension to multi-particle states such that the operator acts on all possible pairs of states:

$$\hat{V}|i_1, \ldots, i_k\rangle = \sum_{l<m} \left(V^{(2)} |i_l\rangle |i_m\rangle \right) \otimes |i_1, \ldots, (\text{no} \;\; i_l), \ldots, (\text{no} \;\; i_m), \ldots, i_k\rangle$$

Can check that

$$\hat{V} = \sum_{ij,kl} V_{ij,kl} a_i^\dagger a_j^\dagger a_k a_l.$$

The field concept

The field is the most important tool for doing multi-particle physics!!!

$$\boxed{\hat{\phi}(\xi) = \sum_i a_i \Psi_i(\xi)}$$

In words: the linear combination of annihilation operators; the coefficients are wave-functions $\Psi_i(\xi)$.

$$[\hat{\phi}(\xi), \hat{\phi}(\xi')] = \delta(\xi, \xi').$$

The operators in term of the field

$$\hat{A} = \int d\xi \hat{\phi}^{\dagger}(\xi) A^{(1)} \hat{\phi}(\xi).$$

$$\hat{V} = \int d\xi d\xi' \hat{\phi}^{\dagger}(\xi) \hat{\phi}^{\dagger}(\xi') V^{(2)} \hat{\phi}(\xi) \hat{\phi}(\xi').$$

Summary of second quantization

- Multi-particle states $|i_1, \ldots, i_k\rangle$, creation-annihilation operators $a_i, a_i^\dagger$, and the field $\hat{\phi}$.

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- Quantum field theory arises by applying the procedure of second quantization to the classical system: the relativistic field.