Lecture 1: Introduction to QFT and Second Quantization

• General remarks about quantum field theory.

• What is quantum field theory about?

• Why relativity plus QM imply an unfixed number of particles?

• Creation-annihilation operators.

• Second quantization.
General remarks

- Quantum Field Theory as the theory of “everything”: all other physics is derivable, except gravity.

- The pinnacle of human thought. The distillation of basic notions from the very beginning of the physics.

- May seem hard but simple and beautiful once understood.
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• The number of particles is **unfixed**.

• QFT can describe relativistic as well as non-relativistic systems.

• QFT’s technical and conceptual difficulties stem from it having to describe processes involving an unfixed number of particles.
Why relativity plus QM implies an unfixed number of particles

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Why relativity plus QM implies an unfixed number of particles

- Relativity says that Mass=Energy.
- Quantum mechanics makes any energy available for a short time: $\Delta E \cdot \Delta t \sim \hbar$
Creation-Annihilation operators

**Problem:** Suppose $[a, a^\dagger] = 1$. Find the spectrum of the hermitian operator $a^\dagger a$. 
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Solution: Eigenvalues are positive integers $n$. The action of $a, a^\dagger$ on normalized eigenstates is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle,$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

In particular, there is the “vacuum” state $|0\rangle : a|0\rangle = 0$.

Interpretation: The number $n$ as the number of “quanta” in the state.
Second Quantization

- Consider an arbitrary quantum or classical system, whose states are denoted by $|i\rangle$. E.g. can be eigenstates of the Hamiltonian:

$$\hat{H} |i\rangle = E_i |i\rangle$$
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- Consider an arbitrary quantum or classical system, whose states are denoted by $|i\rangle$. E.g. can be eigenstates of the Hamiltonian:

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- Want to allow any number of “quanta” or “particles” in each state $|i\rangle$. A general state is specified by its occupation numbers $n_1, n_2, \ldots$:

$$|n_1, n_2, \ldots\rangle$$
Introduce creation-annihilation operators, one copy for each $i$:

$$[a_i, a_i^\dagger] = 1,$$

$$a_i |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i} |n_1, n_2, \ldots, n_i - 1, \ldots\rangle,$$

$$a_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1} |n_1, n_2, \ldots, n_i + 1, \ldots\rangle.$$

The operators $a_i, a_i^\dagger$ create-annihilate particles in state $i$.

The general state can be built from the vacuum:

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \prod_i \frac{(a_i^\dagger)^{n_i}}{(n_i!)^{1/2}} |0\rangle.$$

Here $|0\rangle : a_i |0\rangle = 0$, $\forall i$. 
Multi-particle states: more convenient notation

\[ |i_1, \ldots, i_k\rangle = a_{i_1}^\dagger \ldots a_{i_k}^\dagger |0\rangle \]

If all \( i_1, \ldots, i_k \) are all different - correctly normalized. Otherwise norm greater than one.

Relation between the “occupation number” and the “multi-particle” notations:

\[
\begin{array}{c}
\big| \underbrace{1, \ldots, 1}_{n_1}, \underbrace{2, \ldots, 2}_{n_2}, \ldots \bigangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \ldots |0\rangle = \sqrt{n_1!n_2!\ldots} |n_1, n_2, \ldots\rangle
\end{array}
\]
The action of creation-annihilation operators on multi-particle states:

\[ a_i^\dagger |i_1, \ldots, i_k\rangle = |i, i_1, \ldots, i_k\rangle, \]

\[ a_i |i_1, \ldots, i_k\rangle = \sum_{l=1}^{k} \delta_{ii_l} |i_1, \ldots, (\text{no } i_l), \ldots, i_k\rangle = \sum_{l=1}^{k} \langle i |i_l\rangle |i_1, \ldots, (\text{no } i_l), \ldots, i_k\rangle. \]

**Fock space**: Can form a Hilbert space by considering arbitrary linear combinations of multi-particle states. States with different number of particles are orthogonal.
Multi-particle operators: one particle

**Problem:** Have an operator \(A^{(1)}\) that “knows” how to act on 1-particle states. How to extend it on multi-particle states?

Since \(|i_1, \ldots, i_k\rangle = |i_1\rangle \otimes \ldots \otimes |i_k\rangle\), then natural to define:

\[
\hat{A}|i_1, \ldots, i_k\rangle = A^{(1)}|i_1\rangle \otimes \ldots \otimes |i_k\rangle + \ldots |i_1\rangle \otimes \ldots \otimes A^{(1)}|i_k\rangle.
\]

If each state - eigenstate of \(A^{(1)}\), then

\[
\hat{A}|i_1, \ldots, i_k\rangle = (a_1 + \ldots + a_k)|i_1, \ldots, i_k\rangle.
\]

What about a general operator?
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What about a general operator?

$$ \hat{A} = \sum_{i,j} A_{ij} a_i^\dagger a_j. $$
2-Particle operators

\[ V^{(2)} = \sum_{ij,kl} V_{ij,kl} |ij\rangle \langle kl|. \]

Want to have an extension to multi-particle states such that the operator acts on all possible pairs of states:

\[ \hat{V} |i_1, \ldots, i_k\rangle = \sum_{l<m} \left( V^{(2)} |i_l\rangle |i_m\rangle \right) \otimes |i_1, \ldots, (\text{no } i_l), \ldots, (\text{no } i_m), \ldots, i_k\rangle \]

Can check that

\[ \hat{V} = \sum_{ij,kl} V_{ij,kl} a_i^\dagger a_j^\dagger a_k a_l. \]
The field concept

The field is the most important tool for doing multi-particle physics!!!

\[ \hat{\phi}(\xi) = \sum_i a_i \Psi_i(\xi) \]

In words: the linear combination of annihilation operators; the coefficients are wave-functions \( \Psi_i(\xi) \).

\[ [\hat{\phi}(\xi), \hat{\phi}(\xi')] = \delta(\xi, \xi') . \]
The operators in term of the field

\[ \hat{A} = \int d\xi \hat{\phi}^\dagger(\xi) A^{(1)}(\xi) \hat{\phi}(\xi). \]

\[ \hat{V} = \int d\xi d\xi' \hat{\phi}^\dagger(\xi) \hat{\phi}^\dagger(\xi') V^{(2)}(\xi, \xi') \hat{\phi}(\xi) \hat{\phi}(\xi'). \]
Summary of second quantization

- Multi-particle states $|i_1, \ldots, i_k\rangle$, creation-annihilation operators $a_i, a_i^\dagger$, and the field $\hat{\phi}$. 
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• The formalism of multi-particle states and operators can be greatly developed. Very useful in statistical and solid state physics.
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• The formalism of multi-particle states and operators can be greatly developed. Very useful in statistical and solid state physics.

• Quantum field theory arises by applying the procedure of second quantization to the classical system: the relativistic field.